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FRANK MORLEY, Editor

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WITH THE COÖPERATION OF

CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

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ON THE NUMBER OF SOLUTIONS IN POSITIVE INTEGERS OF THE EQUATION $yz + zx + xy = n$.

BY L. J. MORDELL.

About sixty years ago, Liouville,* in commenting upon a paper by Hermite, stated some results concerning the number of solutions in positive integers of the equation

$$yz + zx + xy = n, \tag{1}$$

where, of course, n is a positive integer. For example, if x, y, z are odd and $y + z \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, the number of solutions is represented by $F(n)$, the number of uneven binary classes of determinant $-n$. Again if $n \equiv 1 \pmod 4$ and y and z are both odd, the number of solutions is $F(n)$ plus one half of the number of divisors of n. The following explanation of the meaning of the symbol $F(n)$ may make these results more intelligible to those who have not studied the theory of numbers. All the quadratic forms

$$a\xi^2 + 2b\xi\eta + c\eta^2$$

usually denoted by (a, b, c) where a, b, c are any integers satisfying

$$b^2 - ac = -n, \qquad a > 0,$$

can be grouped in a finite number of classes such that the forms in a class can be transformed into each other by a linear substitution

$$\xi' = p\xi + q\eta, \qquad \eta' = r\xi + s\eta$$

where p, q, r, s are integers satisfying the equation

$$ps - qr = 1.$$

Moreover, the forms in two different classes can not be transformed into each other by such a substitution. The total number of classes is called $G(n)$. Representatives of these classes are selected in a particular way† and referred to as reduced forms. We call the classes, in which a and c are not simultaneously even, the uneven classes, and denote the number thereof by $F(n)$. In reckoning these class numbers, we adopt the usual convention, that a class $(k, 0, k)$ is reckoned as $\frac{1}{2}$, and a class $(2k, k, 2k)$ as $\frac{1}{3}$.

* *Jour. de maths.*, series 2, tome 7, 1862, page 44.

† See for example Matthew's "Theory of Numbers," pp. 69–73.

Only a few months ago, Prof. E. T. Bell in his paper * "Class Numbers and the Form $yz + zx + xy$" proved that the number of solutions of equation (1) is equal to $3G(n) - 3$ if n is a prime. He stated that his method, which depends upon formulæ of the type introduced by Liouville into Analysis, also gives the results for n composite, but that he has not published them as they are rather complicated.

I shall now show that the number of solutions of equation (1), when no restrictions are made upon x, y, z, n, except that they are all positive integers, is equal to $3G(n)$, provided that a solution in which one of the unknowns is zero is reckoned as $\frac{1}{2}$ instead of 1. For example, if $n = 19$, the number of solutions is 12; six solutions arising from the permutations of 1, 3, 4, three from 1, 1, 9, and three from the six permutations of 0, 1, 19, since each solution is now reckoned as $\frac{1}{2}$. Also there are four classes of binary forms of determinant -19 represented by (1, 0, 19), (2, 1, 10), (4, $\pm$ 1, 5), so that the formula is verified. It of course includes Bell's result as a particular case, since he has not adopted the convention for the solutions with one of the unknowns equal to zero.

Consider separately the solutions for which $x + y$ is odd or even. In the former case put

$$\begin{aligned} 2x &= 2m + 1 + t, \\ 2y &= 2m + 1 - t, \end{aligned}$$

so that t is an odd number and

$$0 < |t| \leq 2m + 1,$$

and m is a positive integer or zero. The equation (1) becomes

$$\begin{aligned} 4n &= (2m + 1)^2 - t^2 + 4z(2m + 1) \\ &= (2m + 1)(2m + 1 + 4z) - t^2, \end{aligned} \tag{2}$$

where $m, z = 0, 1, 2, \cdots,$

$$0 < |t| \leq 2m + 1$$

and the convention applies to the solutions for which either $|t| = 2m + 1$ or $z = 0$. The number of solutions of (2) is equal to $F(4n)$, as we can establish a unique correspondence between them and the uneven classes of binary quadratics of determinant $-4n$. For corresponding to any solution, we have the quadratic form of determinant $-4n$,

$$(2m + 1, \quad t, \quad 2m + 1 + 4z). \tag{3}$$

But the uneven classes of determinant $-4n$ can be represented by the

* *Tôhoku Mathematical Journal*, Vol. 19, May, 1921, pp. 105–116.

forms (a, b, c) with

$$b^2 - ac = -4n,$$

where $c \geq a \geq 2|b|$ and c and a are not both even; if any of the equality signs hold, we take only the positive values of b. These forms can be arranged in three groups according to the residues of $a, b, c \pmod 2$.

In group I, a, b, c are all odd, whence $a \equiv c \pmod 4$. In group II, a is odd, b is even, c is even; and in group III, a is even, b is even, c is odd. Now the number of solutions of (2) in which $|t| \leq m$ is obviously equal to the number of forms in the first group, since when $z = 0$, the solution is reckoned as $\frac{1}{2}$, that is the forms $(2m + 1, t, 2m + 1)$, are reckoned only when t is positive. For the solutions with $|t| > m$ we consider instead of the quadratic (3), the form derived from it, by changing x into $x \mp y$ according as $t >< 0$, namely,

$$[2m + 1, t \mp (2m + 1), (4m + 2 + 4z \mp 2t)] = (A, B, C) \text{ say.} \qquad (4)$$

Hence $A > 2|B|$ and either the form (A, B, C) or $(C, -B, A)$ is reduced, so that the number of solutions now is equal to the number of forms in the groups (2) and (3). We note that when $|t| = 2m + 1$, the convention concerning zero solutions means that the form $(2m + 1, 0, 4z)$ is only reckoned once. This proves that when $x + y$ is odd, the number of solutions of (1) is $F(4n)$ and this is also equal to $2F(n)$.

When $x + y$ is even, we put $x = m + t$, $y = m - t$, so that (1) becomes

$$n = m^2 - t^2 + 2mz, \qquad (5)$$

where

$$0 \leq |t| \leq m, \quad \text{and} \quad m, z = 0, 1, 2, 3, \cdots$$

with the convention when either $z = 0$ or $|t| = m$.

The solutions of this equation can be found in exactly the same way as in (2); but I have already done this in my paper "On Class Relation Formulæ" * and the conventions there adopted concerning the number of solutions are exactly the same as the present ones. Two cases are considered. When m is odd, the number of solutions† is $F(n)$; when m is even, the number of solutions‡ is $3G(n) - 3F(n)$. Hence the total number of solutions of (5) is $3G(n) - 2F(n)$, and adding these to $2F(n)$, the number of solutions of (4), we have the final result, that the number of solutions in positive integers of equation (1) is $3G(n)$ provided we reckon only half the solutions when one of the unknowns is zero.§

* *Messenger of Mathematics*, Vol. 46, 1916.

† Page 133 of the above.

‡ Page 134 of the above.

§ A very simple proof has since been given by Whitehead in the *Proceedings of the London Mathematical Society*, records of proceedings at meetings, etc., issued May 30, 1922.

We have also shown * that

(1) $x + y$ is odd for $2F(n)$ of these solutions,
(2) $x + y \equiv 2 \pmod 4$ for $F(n)$ of these solutions,
(3) $x + y \equiv 0 \pmod 4$ for $3G(n) - 3F(n)$ of these solutions.†

We may note also that when n is not a perfect square

$$x + y \equiv 1 \pmod 4 \text{ for } F(n) \text{ of these solutions,}$$
$$x + y \equiv 3 \pmod 4 \text{ for } F(n) \text{ of these solutions.}$$

For if m is even in equation (2) the binary quadratics (3) represent odd numbers of the form $4k + 1$, and hence are half of the total number of odd forms of determinant $-4n$, as follows from the elementary properties of the generic character of binary quadratic forms.

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* These include Liouville's results.

† Also given by Lerch in the *Rozpravy ceske Akad. Prague*, 7, 1898, No. 4 [Bohemian].

A CLOSED SET OF NORMAL ORTHOGONAL FUNCTIONS.*

By J. L. Walsh.

Introduction.

A set of normal orthogonal functions $\{\chi\}$ for the interval $0 \leqq x \leqq 1$ has been constructed by Haar,† each function taking merely one constant value in each of a finite number of sub-intervals into which the entire interval (0, 1) is divided. Haar's set is, however, merely one of an infinity of sets which can be constructed of functions of this same character. It is the object of the present paper to study a certain new closed set of functions $\{\varphi\}$ normal and orthogonal on the interval (0, 1); each function φ has this same property of being constant over each of a finite number of sub-intervals into which the interval (0, 1) is divided. In fact each function φ takes only the values $+1$ and -1, except at a finite number of points of discontinuity, where it takes the value zero.

The chief interest of the set φ lies in its similarity to the usual (e.g., sine, cosine, Sturm-Liouville, Legendre) sets of orthogonal functions, while the chief interest of the set χ lies in its *dissimilarity* to these ordinary sets. The set φ shares with the familiar sets the following properties, none of which is possessed by the set χ: the nth function has $n - 1$ zeroes (or better, sign-changes) interior to the interval considered, each function is either odd or even with respect to the mid-point of the interval, no function vanishes identically on any sub-interval of the original interval, and the entire set is uniformly bounded.

Each function χ can be expressed as a linear combination of a finite number of functions φ, so the paper illustrates the changes in properties which may arise from a simple orthogonal transformation of a set of functions.

In § 1 we define the set χ and give some of its principal properties. In § 2 we define the set φ and compare it with the set χ. In § 3 and § 4 we develop some of the properties of the set φ, and prove in particular that every continuous function of bounded variation can be expanded in terms of the φ's and that every continuous function can be so developed in the sense not of convergence of the series but of summability by the first Cesàro mean. In § 5 it is proved that there exists a continuous function which

* Presented to the American Mathematical Society, Feb. 25, 1922.

† *Mathematische Annalen*, Vol. 69 (1910), pp. 331–371; especially pp. 361–371.

cannot be expanded in a convergent series of the functions φ. In § 6 there is studied the nature of the approach of the approximating functions to the sum function at a point of discontinuity, and in § 7 there is considered the uniqueness of the development of a function.

§ 1. Haar's Set χ.

Consider the following set of functions:

$$f_0(x) \equiv 1, \qquad 0 \leqq x \leqq 1,$$

$$f_1^{(1)}(x) \equiv \begin{cases} 1, 0 \leqq x < \frac{1}{2}, \\ 0, \frac{1}{2} < x \leqq 1, \end{cases} \qquad f_1^{(2)}(x) \equiv \begin{cases} 1, \frac{1}{2} < x \leqq 1, \\ 0, 0 \leqq x < \frac{1}{2}, \end{cases}$$

. . . .

$$f_k^{(i)}(x) \equiv \begin{cases} 1, \dfrac{i-1}{2^k} < x < \dfrac{i}{2^k}, & i = 1, 2, 3, \cdots, 2^k, \\ 0, 0 \leqq x < \dfrac{i-1}{2^k}, \text{ or } \dfrac{i}{2^k} < x \leqq 1, & k = 1, 2, 3, \cdots, \infty; \end{cases}$$

these functions may be defined at a point of discontinuity to have the average of the limits approached on the two sides of the discontinuity.

If we have at our disposal all the functions $f_k^{(i)}$, it is clear that we can approximate to any continuous function in the interval $0 \leqq x \leqq 1$ as closely as desired and hence that we can expand any continuous function in a uniformly convergent series of functions $f_k^{(i)}$. For a continuous function $F(x)$ is uniformly continuous in the interval (0, 1), and thus uniformly in that entire interval can be approximated as closely as desired by a linear combination of the functions $f_k^{(i)}$ where k is chosen sufficiently large but fixed. The approximation can be made better and better and thus will lead to a uniformly convergent series of functions $f_k^{(i)}$.

Haar's set χ may be found by normalizing and orthogonalizing the set $f_k^{(i)}$, those functions to be ordered with increasing k, and for each k with increasing i. The set χ consists of the following functions:*

$$\chi_0(x) \equiv 1, \qquad 0 \leqq x \leqq 1, \qquad \chi_1(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leqq 1, \end{cases}$$

$$\begin{aligned} \chi_2^{(1)}(x) &= \sqrt{2}, & \chi_2^{(2)} &= 0, & 0 &\leqq x < \tfrac{1}{4}, \\ &= -\sqrt{2}, & &= 0, & \tfrac{1}{4} &< x < \tfrac{1}{2}, \\ &= 0, & &= \sqrt{2}, & \tfrac{1}{2} &< x < \tfrac{3}{4}, \\ &= 0, & &= -\sqrt{2}, & \tfrac{3}{4} &< x \leqq 1, \end{aligned}$$

. . . .

* L. c., p. 361.

$$\begin{aligned} \chi_n^{(k)} &= \sqrt{2^{n-1}}, & \frac{k-1}{2^{n-1}} < x < \frac{2k-1}{2^n}, & \qquad k = 1, 2, 3, \cdots, 2^{n-1}, \\ &= -\sqrt{2^{n-1}}, & \frac{2k-1}{2^n} < x < \frac{k}{2^{n-1}}, & \qquad n = 1, 2, 3, \cdots, \infty, \\ &= 0, & 0 < x < \frac{k-1}{2^{n-1}} \ \text{ or } \ \frac{k}{2^{n-1}} < x < 1. & \end{aligned}$$

The same convention as to the value of $\chi_n^{(k)}$ at a point of discontinuity is made as for the $f_n^{(k)}$, and $\chi_n^{(k)}(0)$ and $\chi_n^{(k)}(1)$ are defined as the limits of $\chi_n^{(k)}$ as x approaches 0 and 1.

For any particular value of N, all the functions $f_n^{(k)}$, $n < N$, can be expressed linearly in terms of the functions $\chi_n^{(k)}$, $n < N$, and conversely.

Let $F(x)$ be any function integrable and with an integrable square in the interval (0, 1); its formal development in terms of the functions χ is

$$\begin{aligned} F(x) \sim \chi_0(x) \int_0^1 F(y)\chi_0(y)dy + \chi_1(x) \int^1 F(y)\chi_1(y)dy + \cdots \\ + \chi_n^{(k)}(x) \int_0^1 F(y)\chi_n^{(k)}(y)dy + \cdots. \end{aligned} \tag{1}$$

This series (1) is formed with coefficients determined formally as for the Fourier expansions, and it is well known that $S_m(x)$, the sum of the first m terms of this series, is that linear combination $F_m(x)$ of the first m of the functions χ which renders a minimum the integral

$$\int_0^1 (F(x) - F_m(x))^2 dx.$$

That is, $S_m(x)$ is in the sense of least squares the best approximation to $F(x)$ which can be formed from a linear combination of the first m functions χ; it is likewise true that $S_m(x)$ is the best approximation to $F(x)$ which can be formed from a linear combination of those functions $f_n^{(k)}$ that are dependent on the first m functions χ.

Let $F(x)$ be continuous in the closed interval (0, 1). If ϵ is any positive number, there exists a corresponding number n such that

$$|F(x') - F(x'')| < \epsilon \qquad \text{whenever} \qquad |x' - x''| < \frac{1}{2^n}\cdot$$

We interpret $S_{2^n}(x)$ as a linear combination of the functions $f_n^{(k)}$. The multiplier of the function $f_n^{(k)}$ which appears in $S_{2^n}(x)$ is chosen so as to furnish the best approximation in the interval $\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ to the function $F(x)$, so it is evident that $S_{2^n}(x)$ approximates to $F(x)$ uniformly in the entire interval (0, 1) with an approximation better than ϵ. The function

$S_{2^{n+1}}(x)$ cannot differ from $F(x)$ by more than ϵ at any point of the interval (0, 1), and so for all the functions $S_{2^{n+i}}(x)$. Thus we have

THEOREM I. *If $F(x)$ is continuous in the interval* (0, 1), *series* (1) *converges uniformly to the value $F(x)$ if the terms are grouped so that each group contains all the 2^{n-1} terms of a set $\chi_n^{(k)}$, $k = 1, 2, 3, \cdots, 2^{n-1}$.*

Haar proves that the series actually converges uniformly to $F(x)$ without the grouping of terms,* and establishes many other results for expansions in terms of the set χ; to some of these results we shall return later.

§ 2. The Set φ.

The set φ, which it is the main purpose of this paper to study, consists of the following functions:

$$\varphi_0(x) \equiv 1, \qquad 0 \leqq x \leqq 1, \qquad \varphi_1(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leqq 1, \end{cases}$$

$$\varphi_2^{(1)}(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{4},\ \frac{3}{4} < x \leqq 1, \\ -1, & \frac{1}{4} < x < \frac{3}{4}, \end{cases}$$

$$\varphi_2^{(2)}(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{4},\ \frac{1}{2} < x < \frac{3}{4}, \\ -1, & \frac{1}{4} < x < \frac{1}{2},\ \frac{3}{4} < x \leqq 1, \end{cases}$$

. . . .

$$\varphi_{n+1}^{(2k-1)}(x) \equiv \begin{cases} \varphi_n^{(k)}(2x), & 0 \leqq x < \frac{1}{2}, \\ (-1)^{k+1}\varphi_n^{(k)}(2x-1), & \frac{1}{2} < x \leqq 1, \end{cases}$$
$$\varphi_{n+1}^{(2k)}(x) \equiv \begin{cases} \varphi_n^{(k)}(2x), & 0 \leqq x < \frac{1}{2}, \\ (-1)^{k}\varphi_n^{(k)}(2x-1), & \frac{1}{2} < x \leqq 1, \end{cases} \tag{2}$$
$$k = 1, 2, 3, \cdots, 2^{n-1}, \qquad n = 1, 2, 3, \cdots, \infty.$$

In general, the function $\varphi_n^{(1)}$, $n > 0$, is to be used, with the horizontal scale reduced one half and the vertical scale unchanged, to form the functions $\varphi_{n+1}^{(1)}$ and $\varphi_{n+1}^{(2)}$ in each of the halves $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$ of the original interval; the function $\varphi_{n+1}^{(1)}(x)$ is to be even and the function $\varphi_{n+1}^{(2)}$ odd with respect to the point $x = \frac{1}{2}$. Similarly the function $\varphi_n^{(k)}$ is to be used to form the functions $\varphi_{n+1}^{(2k-1)}$ and $\varphi_{n+1}^{(2k)}$, the former of which is even and the latter odd with respect to the point $x = \frac{1}{2}$. All the functions $\varphi_n^{(k)}$ are to be taken positive in the interval $\left(0, \dfrac{1}{2^n}\right)$. The function $\varphi_n^{(k)}$ is to be defined at points of discontinuity as were the functions f and χ, and at $x = 0$ to have the value 1, and at $x = 1$ to have the value $(-1)^{k+1}$.† The function

* L. c., p. 368.

† If it is desired to develop periodic functions by means of the set φ [or the similar sets f and χ] simultaneously in all the intervals $\cdots$, $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $\cdots$, it will be wise to change these definitions at $x = 0$ and $x = 1$ so that always the value of $\varphi_n^{(k)}(x)$ is the arithmetic mean of the limits approached at these points to the right and to the left.

$\varphi_n^{(k)}$ is odd or even with respect to the point $x = \frac{1}{2}$ according as k is even or odd.

The functions φ_0, φ_1, $\varphi_2^{(1)}$, $\varphi_2^{(2)}$ have 0, 1, 2, 3 zeroes (i.e., sign-changes) respectively interior to the interval (0, 1). The function $\varphi_{n+1}^{(2k-1)}(x)$ has twice as many zeroes as the function $\varphi_n^{(k)}$; and $\varphi_{n+1}^{(2k)}(x)$ has one more zero, namely at $x = \frac{1}{2}$, than has $\varphi_{n+1}^{(2k-1)}(x)$. Thus the function $\varphi_n^{(k)}$ has $2^{n-1} + k - 1$ zeroes; this formula holds for $n = 2$ and follows for the general case by induction. Hence each function $\varphi_n^{(k)}$ has one more zero than the preceding; the zeroes of these functions increase in number precisely as do the zeroes of the classical sets of functions—sine, cosine, Sturm-Liouville, Legendre, etc. We shall at times find it convenient to use the notation $\varphi_0, \varphi_1, \varphi_2, \cdots$ for the functions $\varphi_n^{(k)}$; the subscript denotes the number of zeroes.

The orthogonality of the system φ is easily established. Any two functions $\varphi_n^{(k)}$ are orthogonal if $n < 3$, as may be found by actually testing the various pairs of functions. Let us assume this fact to hold for $n = 1, 2, 3, \cdots, N - 1$; we shall prove that it holds for $n = N$. By the method of construction of the functions φ, each of the integrals

$$\int_0^{1/2} \varphi_N^{(k)}(x)\varphi_m^{(l)}(x)dx, \qquad \int_{1/2}^1 \varphi_N^{(k)}(x)\varphi_m^{(l)}(x)dx, \qquad m \leqq N,$$

is the same except possibly for sign as an integral

$$\int_0^1 \varphi_{N-1}^{(j)}(y)\varphi_{m-1}^{(l)}(y)dy$$

after the change of variable $y = 2x$ or $y = 2x - 1$. Each of these two integrals [in fact, they are the same integral] whose variable is y has the value zero, so we have the orthogonality of $\varphi_N^{(k)}(x)$ and $\varphi_m^{(l)}(x)$:

$$\int_0^1 \varphi_N^{(k)}(x)\varphi_m^{(l)}(x)dx = 0.$$

This proof breaks down if the two functions $\varphi_{N-1}^{(j)}(y)$, $\varphi_{m-1}^{(l)}(y)$ are the same, but in that case either $\varphi_N^{(k)}(x)$ and $\varphi_m^{(l)}(x)$ are the same and we do not wish to prove their orthogonality, or one of the functions $\varphi_N^{(k)}(x)$, $\varphi_m^{(l)}(x)$ is odd and the other even, so the two are orthogonal.

Each of the functions $\varphi_n^{(k)}(x)$ is normal, for we have

$$|\varphi_n^{(k)}(x)| \equiv 1$$

except at a finite number of points.

Each of the functions $\chi_0, \chi_1, \chi_2^{(1)}, \chi_2^{(2)}, \cdots, \chi_{n+1}^{(2^n)}$ can be expressed linearly in terms of the functions $\varphi_0, \varphi_1, \varphi_2^{(1)}, \varphi_2^{(2)}, \cdots, \varphi_{n+1}^{(2^n)}$. Thus for $n = 1$ we have

$$\chi_0 = \varphi_0, \quad \chi_1 = \varphi_1, \quad \chi_2^{(1)} = \tfrac{1}{2}\sqrt{2}(\varphi_2^{(1)} + \varphi_2^{(2)}), \quad \chi_2^{(2)} = \tfrac{1}{2}\sqrt{2}(-\varphi_2^{(1)} + \varphi_2^{(2)}).$$

It is true generally that except for a constant normalizing factor $\sqrt{2}$, the function $\chi_{n+1}^{(k)}$, $k \leqq 2^{n-1}$, is the same linear combination of the functions $\frac{1}{2}[\varphi_{n+1}^{(2k-1)} + \varphi_{n+1}^{(2k)}]$ as is $\chi_n^{(k)}$ of the functions $\varphi_n^{(k)}$, and the function $\chi_{n+1}^{(k)}$, $k > 2^{n-1}$, is the same linear combination of the functions $\frac{1}{2}(-1)^{k+1}[\varphi_{n+1}^{(2k-1)} - \varphi_{n+1}^{(k)}]$ as is $\chi_n^{(k-2^{n-1})}$ of the functions $\varphi_n^{(k)}$.

It is similarly true that all the functions $\varphi_0, \varphi_1, \cdots, \varphi_{n+1}^{(2^n)}$ can be expressed linearly in terms of the functions $\chi_0, \chi_1, \cdots, \chi_{n+1}^{(2^n)}$. Thus we have for $n = 2$,

$$\varphi_0 = \chi_0, \qquad \varphi_1 = \chi_1, \qquad \varphi_2^{(1)} = \tfrac{1}{2}\sqrt{2}(\chi_2^{(1)} - \chi_2^{(2)}), \qquad \varphi_2^{(2)} = \tfrac{1}{2}\sqrt{2}(\chi_2^{(1)} + \chi_2^{(2)}).$$

The general fact appears by induction from the very definition of the functions φ.

The set χ is known to be closed;* it follows from the expression of the χ in terms of the φ that the set φ is also closed.

The definition of the functions $\varphi_n^{(k)}$ enables us to give a formula for $\varphi_n^{(k)}(x)$. Let us set, in binary notation,

$$x = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots, \qquad a_i = 0 \text{ or } 1.$$

If x is a binary irrational or if in the binary expansion of x there exists $a_i \neq 0$, $i > n$, the following formulas hold for $\varphi_n^{(k)}$:

$$\begin{array}{ll}
\varphi_0 = 1, & \varphi_1 = (-1)^{a_1}, \\
\varphi_2^{(1)} = (-1)^{a_1+a_2}, & \varphi_2^{(2)} = (-1)^{a_2}, \\
\varphi_3^{(1)} = (-1)^{a_2+a_3}, & \varphi_3^{(2)} = (-1)^{a_1+a_2+a_3}, \\
\varphi_3^{(3)} = (-1)^{a_1+a_3}, & \varphi_3^{(4)} = (-1)^{a_3}, \\
\varphi_4^{(1)} = (-1)^{a_3+a_4}, & \varphi_4^{(2)} = (-1)^{a_1+a_3+a_4}, \\
\varphi_4^{(3)} = (-1)^{a_1+a_2+a_3+a_4}, & \varphi_4^{(4)} = (-1)^{a_2+a_3+a_4}, \\
\varphi_4^{(5)} = (-1)^{a_2+a_4}, & \varphi_4^{(6)} = (-1)^{a_1+a_2+a_4}, \\
\varphi_4^{(7)} = (-1)^{a_1+a_4}, & \varphi_4^{(8)} = (-1)^{a_4}, \\
\cdots\cdots & \cdots\cdots, \\
\cdots\cdots & \cdots\cdots.
\end{array} \tag{3}$$

The general law appears from these relations; always we have

$$\begin{aligned} \varphi_n^{(1)} &= (-1)^{a_{n-1}+a_n}, \\ \varphi_n^{(k)} &= \varphi_{k-1}\varphi_n^{(1)}. \end{aligned} \tag{4}$$

A general expression for $\varphi_n^{(k)}(x)$ when x is a binary rational can readily be computed from formulas (3), for we have expressions for the values of $\varphi_n^{(k)}$ for neighboring larger and smaller values of the argument than x.

* That is, there exists no non-null Lebesgue-integrable function on the interval (0, 1) which is orthogonal to all functions of the set; l. c., p. 362.

§ 3. Expansions in Terms of the Set $\{\varphi\}$.

The following theorem results from Theorem I by virtue of the remark that all the functions $\varphi_n^{(k)}$ can be expressed in terms of the functions $\chi_n^{(i)}$ and conversely, and from the least squares interpretation of a partial sum of a series of orthogonal functions:

THEOREM II. *If $F(x)$ is continuous in the interval $(0, 1)$, the series*

$$F(x) \sim \varphi_0(x)\int_0^1 F(y)\varphi_0(y)dy + \varphi_1(x)\int_0^1 F(y)\varphi_1(y)dy + \cdots \varphi_i^{(j)}(x)\int_0^1 F(y)\varphi_i^{(j)}(y)dy + \cdots, \tag{5}$$

converges uniformly to the value $F(x)$ if the terms are grouped so that each group contains all the 2^{n-1} terms of a set $\varphi_n^{(k)}$, $k = 1, 2, 3, \cdots, 2^{n-1}$.

Series (5) after the grouping of terms is precisely the same as series (1) after the grouping of terms.

Theorem II can be extended to include even discontinuous functions $F(x)$; we suppose $F(x)$ to be integrable in the sense of Lebesgue. Let us introduce the notation

$$F(a+0) = \lim_{\epsilon=0} F(a+\epsilon), \qquad F(a-0) = \lim_{\epsilon=0} F(a-\epsilon), \qquad \epsilon > 0,$$

and suppose that these limits exist for a particular point $x = a$. We introduce the functions

$$F_1(x) = \begin{cases} F(x), & x < a, \\ F(a-0), & x \geqq a, \end{cases} \qquad F_2(x) = \begin{cases} F(a+0), & x \leqq a, \\ F(x), & x > a, \end{cases} \tag{6}$$

The least squares interpretation of the partial sums $S_{2^n}(x)$ of the series (1) or (5) as expressed in terms of the $f_i^{(j)}$ gives the result that if $h_1 < F(x) < h_2$ in any interval, then also $h_1 < S_{2^n}(x) < h_2$ in any completely interior interval if n is sufficiently large. It follows that $F_1(x)$ is closely approximated at $x = a$ by its partial sum S_{2^n} if n is sufficiently large, and that this approximation is uniform in any interval about the point $x = a$ in which $F_1(x)$ is continuous. A similar result holds for $F_2(x)$.

The function $F_1(x) + F_2(x)$ differs from the original function $F(x)$ merely by the function

$$G(x) = \begin{cases} F(a+0), & x < a, \\ F(a-0), & x > a. \end{cases}$$

The representation of such functions by sequences of the kind we are considering will be studied in more detail later (§ 6), but it is fairly obvious that such a function is represented uniformly except in the neighborhood

of the point a. If $F(x)$ is continuous at and in the neighborhood of a, or if a is dyadically rational, the approximation to $G(x)$ is uniform at the point a as well. Thus we have

THEOREM III. *If $F(x)$ is any integrable function and if $\lim_{x=a} F(x)$ exists for a point a, then when the terms of the series* (5) *are grouped as described in Theorem II, the series so obtained converges for $x = a$ to the value $\lim_{x=a} F(x)$. If $F(x)$ is continuous at and in the neighborhood of a, then this convergence is uniform in a neighborhood of a.*

If $F(x)$ is any integrable function and if the limits $F(a - 0)$ and $F(a + 0)$ exist for a dyadically rational point $x = a$, then the series with the terms grouped converges for $x = a$ to the value $\frac{1}{2}[F(a + 0) + F(a - 0)]$; this convergence is uniform in the neighborhood of the point $x = a$ if $F(x)$ is continuous on two intervals extending from a, one in each direction.

It is now time to study the convergence of series (5) when the terms are not grouped as in Theorems II and III. We shall establish

THEOREM IV. *Let the function $F(x)$ be of limited variation in the interval $0 \leqq x \leqq 1$. Then the series* (5) *converges to the value $F(x)$ at every point at which $F(a + 0) = F(a - 0)$ and at every point at which $x = a$ is dyadically rational. This convergence is uniform in the neighborhood of $x = a$ in each of these cases if $F(x)$ is continuous in two intervals extending from a, one in each direction.*

Since $F(x)$ is of limited variation, $F(a + 0)$ and $F(a - 0)$ exist at every point a. Theorem IV tacitly assumes $F(x)$ to be defined at every point of discontinuity a so that $F(a) = \frac{1}{2}[F(a + 0) + F(a - 0)]$.

Any such function $F(x)$ can be considered as the difference of two monotonically increasing functions, so the theorem will be proved if it is proved merely for a monotonically increasing function. We shall assume that $F(x)$ is such a function, and positive. We are to evaluate the limit of

$$\int_0^1 F(y) K_n^{(k)}(x, y) dy,$$

$$K_n^{(k)}(x, y) = \varphi_0(x)\varphi_0(y) + \varphi_1(x)\varphi_1(y) + \cdots + \varphi_n^{(k)}(x)\varphi_n^{(k)}(y).$$

We have already evaluated this limit for the sequence $k = 2^{n-1}$, so it remains merely to prove that

$$\lim_{n=\infty} \int_0^1 F(y) Q_n^{(k)}(x, y) dy = 0, \tag{7}$$

$$Q_n^{(k)}(x, y) = \varphi_n^{(1)}(x)\varphi_n^{(1)}(y) + \varphi_n^{(2)}(x)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(k)}(x)\varphi_n^{(k)}(y),$$

whatever may be the value of k.

We shall consider the function $F(x)$ merely at a point $x = a$ of con-

tinuity; that is, we study essentially the new functions F_1 and F_2 defined by equations (6). In the sequel we suppose a to be dyadically irrational; the necessary modifications for a rational can be made by the reader.

The following formulas are easily found by the definition of the $Q_n^{(k)}$; both x and y are supposed dyadically irrational:

$$Q_2^{(1)}(x, y) = \pm 1,$$

$$Q_2^{(2)}(x, y) = \begin{cases} 0 \text{ if } x < \tfrac{1}{2},\ y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\ y < \tfrac{1}{2}, \\ \pm 2 \text{ if } x < \tfrac{1}{2},\ y < \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\ y > \tfrac{1}{2}, \end{cases}$$

. . . .

$$Q_n^{(1)}(x, y) = \pm 1,$$

$$Q_n^{(2)}(x, y) = \begin{cases} 0 \text{ if } x < \tfrac{1}{2},\ y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\ y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(1)}(2x, 2y) \text{ if } x < \tfrac{1}{2},\ y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(1)}(2x - 1, 2y - 1) \text{ if } x > \tfrac{1}{2},\ y > \tfrac{1}{2}, \end{cases}$$

. . . .

$$Q_n^{(2k)}(x, y) = \begin{cases} 0 \text{ if } x < \tfrac{1}{2},\ y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\ y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(k)}(2x, 2y) \text{ if } x < \tfrac{1}{2},\ y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(k)}(2x - 1, 2y - 1) \text{ if } x > \tfrac{1}{2},\ y > \tfrac{1}{2}, \end{cases}$$

$$Q_n^{(2k+1)}(x, y) = \begin{cases} \pm 1 \text{ if } x < \tfrac{1}{2},\ y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\ y < \tfrac{1}{2}, \\ \dfrac{Q_n^{(2k)} + Q_n^{(2k+2)}}{2} \text{ if } x < \tfrac{1}{2},\ y < \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\ y > \tfrac{1}{2}. \end{cases}$$

The integral in (7) for $x = a$ is to be divided into three parts. Consider an interval bounded by two points of the form $x = \dfrac{\rho}{2^\nu}$, $x = \dfrac{\rho + 1}{2^\nu}$, where ρ and ν are integers and such that

$$\frac{\rho}{2^\nu} < a < \frac{\rho + 1}{2^\nu}.$$

Then we have

$$\int_0^1 F_1(y) Q_n^{(k)}(a, y)\,dy = \int_0^{\rho/2^\nu} F_1(y) Q_n^{(k)}(a, y)\,dy$$
$$+ \int_{\rho/2^\nu}^{(\rho+1)/2^\nu} F_1(y) Q_n^{(k)}(a, y)\,dy + \int_{(\rho+1)/2^\nu}^1 F_1(y) Q_n^{(k)}(a, y)\,dy. \tag{8}$$

These integrals on the right need separate consideration.

Let us set

$$\frac{\rho}{2^\nu} = \frac{\mu_1}{2^1} + \frac{\mu_2}{2^2} + \frac{\mu_3}{2^3} + \cdots + \frac{\mu_\nu}{2^\nu}, \qquad \mu_i = 0 \text{ or } 1.$$

The first integral in the right-hand member of (8) can be written

$$\int_0^{\mu_1/2^1} + \int_{\mu_1/2^1}^{(\mu_1/2^1)+(\mu_2/2^2)} + \cdots \int_{(\rho/2^\nu)-(\mu_\nu/2^\nu)}^{\rho/2^\nu} F_1(y) Q_n^{(k)}(a, y)\,dy. \tag{9}$$

Each of these integrals is readily treated. Thus, on the interval $0 \leqq y \leqq \frac{\mu_1}{2^1}$, $Q_n^{(k)}(a, y)$ takes only the values ± 1 or 0, is 0 if k is even and has the value $\pm \varphi_n^{(k)}(y)$ if k is odd. It is of course true that

$$\lim_{n=\infty} \int_0^1 \Phi(y)\varphi_n^{(k)}(y)dy = 0 \tag{10}$$

no matter what may be the function $\Phi(y)$ integrable in the sense of Lebesgue and with an integrable square.* Hence we have

$$\lim_{n=\infty} \int_0^{\mu_1/2^1} F_1(y)Q_n^{(k)}(a, y)dy = 0.$$

On the interval $\frac{\mu_1}{2^1} \leqq y \leqq \frac{\mu_1}{2^1} + \frac{\mu_2}{2^2}$, the function $Q_n^{(k)}(a, y)$ takes only the values $0, \pm 1, \pm 2$, and except for one of these numbers as constant factor, has the value $\varphi_n^{(k)}(y)$. It is thus true that

$$\lim_{n=\infty} \int_{\mu_1/2^1}^{(\mu_1/2^1)+(\mu_2/2^2)} F_1(y)Q_n^{(k)}(a, y)dy = 0.$$

From the corresponding result for each of the integrals in (9) and a similar treatment of the last integral in the right-hand member of (8), we have

$$\begin{aligned} &\lim_{n=\infty} \int_0^{\rho/2^\nu} F_1(y)Q_n^{(k)}(a, y)dy = 0, \\ &\lim_{n=\infty} \int_{(\rho+1)/2^\nu}^1 F_1(y)Q_n^{(k)}(a, y)dy = 0. \end{aligned} \tag{11}$$

We shall obtain an upper limit for the second integral in (8) by the second law of the mean. We notice that

$$\left| \int_\xi^{(\rho+1)/2^\nu} Q_n^{(k)}(a, y)dy \right| \leqq \tfrac{1}{2},$$

whatever may be the value of ξ. In fact, this relation is immediate if n

* This well-known fact follows from the convergence of the series

$$\Sigma\, (a_n^{(k)})^2,$$

proved from the inequality

$$\int_0^1 (\Phi(x) - a_0\varphi_0 - a_1\varphi_1 - a_2^{(1)}\varphi_2^{(1)} - \cdots - a_n^{(k)}\varphi_n^{(k)})^2 dx \geqq 0,$$

where $a_n^{(k)} = \int_0^1 \Phi(y)\varphi_n^{(k)}(y)dy$.

is small and it follows for the larger values of n by virtue of the method of construction of the $Q_n^{(k)}$. Moreover, if $n \geqq \nu$ and if $\xi = \frac{\rho}{2^\nu}$, this integral has the value zero. We therefore have from the second law of the mean, $n \geqq \nu$,

$$\int_{\rho/2^\nu}^{(\rho+1)/2^\nu} F_1(y) Q_n^{(k)}(a, y)dy = F_1\left(\frac{\rho}{2^\nu}\right)\int_{\rho/2^\nu}^{\xi} Q_n^{(k)}(a, y)dy$$
$$+ F_1\left(\frac{\rho+1}{2^\nu}\right)\int_{\xi}^{(\rho+1)/2^\nu} Q_n^{(k)}(a, y)dy$$
$$= \left[F_1(a) - F_1\left(\frac{\rho}{2^\nu}\right)\right]\int_{\xi}^{(\rho+1)/2^\nu} Q_n^{(k)}(a, y)dy.$$

By a proper choice of the point $\frac{\rho}{2^\nu}$ we can make the factor of this last integral as small as desired; the entire expression will be as small as desired for sufficiently large n. The relations (11) are independent of the choice of $\frac{\rho}{2^\nu}$, so (7) is completely proved for the function F_1. A similar proof applies to F_2, so (7) can be considered as completely proved for the original function $F(x)$.

The uniform convergence of (5) as stated in Theorem IV follows from the uniform continuity of $F(x)$ and will be readily established by the reader.

§ 4. Further Expansion Properties of the Set φ.

The least square interpretation already given for the partial sums and the expression of the φ's in terms of the f's show that if the terms of (5) are grouped as in Theorems II and III, the question of convergence or divergence of the series at a point depends merely on that point and the nature of the function $F(x)$ in the neighborhood of that point. This same fact for series (5) when the terms are not grouped follows from (8) and (10) if $F(x)$ is integrable and with an integrable square. We shall further extend this result and prove:

THEOREM V. *If $F(x)$ is any integrable function, then the convergence or divergence of the series* (5) *at a point depends merely on that point and on the behavior of the function in the neighborhood of that point. If in particular $F(x)$ is of limited variation in the neighborhood of a point $x = a$, and if a is dyadically rational or if $F(a - 0) = F(a + 0)$, then series* (5) *converges for $x = a$ to the value $\frac{1}{2}[F(a - 0) + F(a + 0)]$. If $F(x)$ is not only of limited variation but is also continuous in two neighborhoods one on each side of a, and if a is dyadically rational or if $F(a - 0) = F(a + 0)$, the convergence of* (5) *is uniform in the neighborhood of the point a.*

Theorem V follows immediately from the reasoning already given and from (10) proved without restriction on Φ; we state the theorem for any bounded normal orthogonal set of functions ψ_n:

THEOREM VI. *If $\{\psi_n(x)\}$ is a uniformly bounded set of normal orthogonal functions on the interval* (0, 1), *and if $\Phi(x)$ is any integrable function, then*

$$\lim_{n=\infty} \int_0^1 \Phi(x)\psi_n(x)dx = 0. \tag{12}$$

Denote by E the point set which contains all points of the interval for which $|\Phi(x)| > N$; we choose N so large that

$$\int_E |\Phi(x)|dx < \epsilon,$$

where ϵ is arbitrary. Denote by E_1 the point set complementary to E; then we have

$$\int_0^1 \Phi(x)\psi_n(x)dx = \int_E \Phi(x)\psi_n(x)dx + \int_{E_1} \Phi(x)\psi_n(x)dx.$$

It follows from the proof of (10) already indicated that the second integral on the right approaches zero as n becomes infinite. The first integral is in absolute value less than $M\epsilon$ whatever may be the value of n, where M is the uniform bound of the ψ_n. It therefore follows that these two integrals can be made as small as desired, first by choosing ϵ sufficiently small and then by choosing n sufficiently large.*

It is interesting to note that Theorem VI breaks down if we omit the hypothesis that the set ψ_n is uniformly bounded. In fact Theorem VI does not hold for Haar's set χ. Thus consider the function

$$\Phi(x) = (x - \tfrac{1}{2})^{-\nu}, \qquad \nu < 1.$$

We have

$$\int_0^1 \Phi(x)\chi_n^{(2^{n-2}+1)}(x)dx = \sqrt{2^{n-1}} \int_{1/2}^{1/2+1/2^n} (x - \tfrac{1}{2})^{-\nu}dx$$
$$- \sqrt{2^{n-1}} \int_{1/2+1/2^n}^{1/2+1/2^{n-1}} (x - \tfrac{1}{2})^{-\nu}dx = \frac{(2^{n-1})^{\nu-(1/2)}}{1-\nu}[2^\nu - 1].$$

Whenever $\nu \geqq \frac{1}{2}$, it is clear that (12) cannot hold, and if $\nu > \frac{1}{2}$, there is a sub-sequence of the sequence in (12) which actually becomes infinite.

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* Theorem VI is proved by essentially this method for the set $\psi_n(x) = \sqrt{2} \sin n\pi x$ by Lebesgue, *Annales scientifiques de l'école normale supérieure*, ser. 3, Vol. XX, 1903. See also Hobson, *Functions of a Real Variable* (1907), p. 675, and Lebesque, *Annales de la Faculté des Science de Toulouse*, ser 3, Vol. I (1909), pp. 25–117, especially p. 52.

We turn now from the study of the convergence of such a series expansion as (5) to the study of the summability of such expansions, and are to prove

THEOREM VII. *If $F(x)$ is continuous in the closed interval* (0, 1), *the series* (5) *is summable uniformly in the entire interval to the sum $F(x)$.*

If $F(x)$ is integrable in the interval (0, 1), *and if $F(a-0)$ and $F(a+0)$ exist, and if either $F(a-0) = F(a+0)$ or a is dyadically rational, then the series* (5) *is summable for $x = a$ to the value $\frac{1}{2}[F(a-0) + F(a+0)]$. If $F(x)$ is continuous in the neighborhood of the point $x = a$, or if a is dyadically rational and $F(x)$ continuous in the neighborhood of a except for a finite jump at a, the summability is uniform throughout a neighborhood of that point.*

In this theorem and below, the term *summability* indicates summability by the first Cesàro mean.

We shall find it convenient to have for reference the following

LEMMA. *Suppose that the series*

$$\begin{aligned}(b_1 + b_2 + \cdots + b_{n_1}) + (b_{n_1+1} + b_{n_1+2} + \cdots + b_{n_2}) + \cdots \\ + (b_{n_k+1} + b_{n_k+2} + \cdots + b_{n_{k+1}}) + \cdots\end{aligned} \tag{13}$$

converges to the sum B and that the sequence

$$\begin{gathered} b_1,\quad \frac{2b_1 + b_2}{2},\quad \frac{3b_1 + 2b_2 + b_3}{3},\quad \cdots \\ \frac{(n_1-1)b_1 + (n_1-2)b_2 + \cdots + b_{n_1-1}}{n_1 - 1}, \\ \frac{(n_1-1)b_1 + \cdots + b_{n_1-1}}{n}, \\ \frac{(n_1-1)b_1 + (n_1-2)b_2 + \cdots + b_{n_1-1} + b_{n_1+1}}{n_1+1}, \\ \frac{(n_1-1)b_1 + \cdots + b_{n_1-1} + 2b_{n_1+1} + b_{n_1+2}}{n_1+2},\quad \cdots \\ \frac{(n_1-1)b_1 + \cdots + b_{n_1-1} + (n_2 - n_1 - 1)b_{n_1+1} + (n_2 - n_1 - 2)b_{n_1+2} + \cdots + b_{n_2-1}}{n_2 - 1}, \\ \cdots, \end{gathered} \tag{14}$$

converges to zero. Then the series

$$b_1 + b_2 + b_3 + \cdots \tag{15}$$

is summable to the sum B.

This lemma involves merely a transformation of the formulas involving the limit notions. Insert zeroes in series (13) so that the parentheses are respectively the n_1-th, n_2-th, n_3-th terms of the new series; this new series

converges to the sum B and hence is summable to the sum B. The term-by-term difference of the new series and (15) is the series

$$b_1 + b_2 + \cdots + b_{n_1-1} - (b_1 + b_2 + \cdots + b_{n_1-1}) + b_{n_1+1} + b_{n_1+2} + \cdots + b_{n_2-1} - (b_{n_1+1} + b_{n_1+2} + \cdots + b_{n_2-1}) + \cdots, \tag{16}$$

which is to be shown to be summable to the sum zero. The sequence corresponding to the summation of (16) is precisely (14).

A sufficient condition for the convergence to zero of (14) is that we have, independently of m,

$$\lim_{k=\infty} \frac{mb_{n_k+1} + (m-1)b_{n_k+2} + \cdots + b_{n_k+m}}{m} = 0, \qquad m \leqq n_{k+1} - n_k, \tag{17}$$

for from a geometric point of view each term of the sequence (14) is the center of gravity of a number of terms such as occur in (17), each term weighted according to the number of b_i that appear in it. An (ϵ, δ)-proof can be supplied with no difficulty.

For the case of Theorem VII let us assume $F(x)$ integrable and that $F(a-0)$ and $F(a+0)$ exist. The series (15) is to be identified with the series (5), and (13) with (5) after the terms are grouped as in Theorem III. The sum that appears in (17) is, then, for $x = a$,

$$\frac{1}{m}\int_0^1 [m\varphi_n^{(1)}(a)\varphi_n^{(1)}(y) + (m-1)\varphi_n^{(2)}(a)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(m)}(a)\varphi_n^{(m)}(y)]F(y)dy, \quad m \leqq 2^{n-1}. \tag{18}$$

We shall prove that (18) formed for the function $F_1(y)$ defined in (6) and for a dyadically irrational has the limit zero as n becomes infinite.

Let us notice that

$$\frac{1}{m}\int_0^1 |m\varphi_n^{(1)}(a)\varphi_n^{(1)}(y) + (m-1)\varphi_n^{(2)}(a)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(m)}(a)\varphi_n^{(m)}(y)|dy = 1. \tag{19}$$

This follows directly from (3) and (4). The value of the integral in (19) is unchanged if we replace a by any dyadic irrational b. Choose $0 < b < 2^{-n}$, so that all the functions $\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{m-1}$ are positive for $x = b$. Then the integrand in (19) can be reduced merely to $m\varphi_0(y)$, so (19) is proved.

Let us consider the integral (18) formed for the function $F_1(y)$ to be divided as in (8), where as before

$$\frac{\rho}{2^\nu} < a < \frac{\rho+1}{2^\nu},$$

and let us denote by (20), (21), (22), (23) respectively the entire integral and its three parts. Then (22) can be made as small as desired simply by proper choice of the point $\frac{\rho}{2^\nu}$, for in the interval $\left(\frac{\rho}{2^\nu}, \frac{\rho+1}{2^\nu}\right)$ we can make $|F_1(y) - F_1(a)|$ uniformly small, we have established (19), and we have also

$$\int_{\rho/2^\nu}^{(\rho+1)/2^\nu} [m\varphi_n^{(1)}(a)\varphi_n^{(1)}(y) + (m-1)\varphi_n^{(2)}(a)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(m)}(a)\varphi_n^{(m)}(y)]F_1(a)dy = 0$$

if merely $n > \nu$.

The integral (21) is the average of m integrals of the type that appear in (8):

$$\int_0^{\rho/2^\nu} F_1(y)Q_n^{(k)}(a, y)dy, \qquad k = 1, 2, \cdots, m.$$

Thus the entire integral (21) approaches zero as n becomes infinite. Treatment in a similar way of the integral (23) proves that (20) approaches zero. It is likewise true that (18) formed for the function $F_2(y)$ also approaches zero as n becomes infinite. This completes the proof of the second sentence in Theorem VII for a dyadic irrational; we omit the proof for a dyadic rational. The uniformity of the continuity of $F(x)$ gives us readily the remaining parts of Theorem VII.

§ 5. Not Every Continuous Function Can Be Expanded in Terms of the φ.

The summability of the expansions of continuous functions in terms of the functions φ is another point of resemblance of those functions to the Fourier sine and cosine functions. Still another point of resemblance which we shall now establish is that there exists a continuous function whose expansion in terms of the φ's does not converge at every point of the interval.

Our proof rests on a beautiful theorem due to Haar,* by virtue of which the existence of such a continuous function will be shown if we prove merely that

$$\int_0^1 |K_n^{(k)}(a, y)|dy \tag{24}$$

is not bounded uniformly for all n and k. The point a is a point of divergence of the expansion of the continuous function and for our particular case may be chosen any point of the interval (0, 1). We shall study (24) in detail merely for a dyadically irrational; the integral (24) is independent of the point a chosen if a is dyadically irrational.

* L. c., p. 335. This condition holds for any set of normal orthogonal functions and is necessary as well as sufficient, if a slight restriction is added.

The integral (24) is bounded uniformly for all the values n if $k = 2^{n-1}$, so it will be sufficient to consider the integral

$$c_n^{(k)} = \int_0^1 |Q_n^{(k)}(a, y)|\, dy.$$

The following table shows the value of $c_n^{(k)}$ for small values of n and for each value of k:

$n = 2$	1	1														
$n = 3$	1	1	$1\frac{1}{2}$	1												
$n = 4$	1	1	$1\frac{1}{2}$	1	$1\frac{3}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	1								
$n = 5$	1,	1,	$1\frac{1}{2}$,	1,	$1\frac{3}{4}$,	$1\frac{1}{2}$,	$1\frac{3}{4}$,	1,	$1\frac{7}{8}$,	$1\frac{3}{4}$,	$2\frac{1}{8}$,	$1\frac{1}{2}$,	$2\frac{1}{8}$,	$1\frac{3}{4}$,	$1\frac{7}{8}$,	1,
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

We have the general formulas

$$c_n^{(1)} = c_n^{(2n+1)} = 1,$$
$$c_n^{(k)} = c_{n+1}^{(2k)},$$
$$c_{n+1}^{(2k+1)} = \tfrac{1}{2}[c_n^{(k)} + c_n^{(k+1)}] + \tfrac{1}{2},$$

so the $c_n^{(k)}$ are not uniformly bounded.

Theorem VIII. *If a point a is arbitrarily chosen, there will exist a continuous function whose φ-development does not converge at a.*

§ 6. The Approximation to a Function at a Discontinuity.

We have considered in § 3 and § 4 with a fair degree of completeness the nature of the approach to $F(x)$ of the formal development of an arbitrary function $F(x)$ in the neighborhood of a point of continuity of $F(x)$. We shall now consider the approach to $F(x)$ of this formal development in the neighborhood of a point of discontinuity of $F(x)$. We study this problem merely for a function which is constant except for a single discontinuity, a finite jump, but this leads directly to similar results for any function $F(x)$ at an isolated discontinuity which is a finite jump, if $F(x)$ is of such a nature that the expansion of $F(x)$ would converge uniformly in the neighborhood of the point of discontinuity were that discontinuity removed by the addition of a function constant except for a finite jump.

Let us consider the function

$$f(x) = \begin{cases} 1, & 0 \leqq x < a, \\ 0, & a < x \leqq 1. \end{cases}$$

If a is dyadically rational, $f(x)$ can be expressed as a finite sum of functions φ,* and thus is represented uniformly, if we make the definition $f(a)$

* A discontinuity at $x = 0$ or $x = 1$ is slightly different [compare the first footnote of § 2]. Under the present definition of the φ's it acts like an artificial discontinuity in the interior of the interval and has no effect on the sequence representing the function.

$= \frac{1}{2}[f(a-0)+f(a+0)]$; this follows from the evident possibility of expanding $f(x)$ in terms of the functions $f_0, f_1, f_2^{(1)}, \cdots$.

If the point a is dyadically irrational, $f(x)$ *cannot be expanded in terms of the* φ. The formal development of $f(x)$ converges in fact for every value of x other than a and diverges for $x = a$.* The convergence for $x \neq a$ follows, indeed, from Theorem IV. We proceed to demonstrate the divergence.

Use the dyadic notation

$$a = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots, \qquad a_n = 0 \text{ or } 1.$$

The partial sum

$$S_n^{(k)}(x) = \varphi_0(x)\int_0^1 f(y)\varphi_0(y)dy + \varphi_1(x)\int_0^1 f(y)\varphi_1(y)dy + \cdots + \varphi_n^{(k)}(x)\int_0^1 f(y)\varphi_n^{(k)}(y)dy$$

is in the sense of least squares the best approximation to $f(x)$ that can be formed from the functions $\varphi_0, \varphi_1, \cdots, \varphi_n^{(k)}$. It is therefore true that when $k = 2^{n-1}$, on every subinterval $\left(\frac{r}{2^n}, \frac{r+1}{2^n}\right)$ on which $f(x)$ is constant, $S_n^{(k)}(x)$ is also constant and equal to $f(x)$. On that subinterval $\left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$ which contains the point a, $S_n^{(k)}$ has the value

$$2^n a - m = \frac{a_{n+1}}{2^1} + \frac{a_{n+2}}{2^2} + \frac{a_{n+3}}{2^3} + \cdots, \tag{25}$$

which lies between zero and unity. Thus $S_n^{(k)}(x)$ $[n > 1]$ is a function with two points of discontinuity and which takes on three distinct values at its totality of points of continuity.

The infinite series corresponding to the sequence (25) is

$$\begin{aligned}\left(\frac{a_2}{2^1} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \cdots\right) &+ \left(\frac{a_3}{2^2} + \frac{a_4}{2^3} + \cdots - \frac{a_2}{2}\right) \\ &+ \left(\frac{a_4}{2^2} + \frac{a_5}{2^3} + \frac{a_6}{2^4} + \cdots - \frac{a_3}{2}\right) \\ &+ \left(\frac{a_5}{2^2} + \frac{a_6}{2^3} + \frac{a_7}{2^4} + \cdots - \frac{a_4}{2}\right) + \cdots.\end{aligned} \tag{26}$$

Not all the numbers a_n after a certain point can be zero and not all of them

* This was pointed out for the set χ by Faber, *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. 19 (1910), pp. 104–112.

can be unity, so the general term of the series (26) cannot approach zero and the sequence (25) cannot converge.

It is likewise true that the sequence (25) is not always summable and if summable may not be summable to the value $\frac{1}{2}$. Thus if we choose

$$a = \frac{1}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{1}{2^7} + \cdots,$$

the sequence (25) is summable to the sum $\frac{2}{3}$. Likewise the sequence $S_n^{(k)}(x)$ for $x = a$ and where we consider all values of n and k, is summable to the value $\frac{2}{3}$.

The general behavior of $S_n^{(k)}(x)$ for $f(x)$ where we do not make the restriction $k = 2^{n-1}$ is quite easily found from the behavior for $k = 2^{n-1}$ and the relation

$$\varphi_n^{(i)}(a) \int_0^1 f(y)\varphi_n^{(i)}(y)dy = \varphi_n^{(k)}(a) \int_0^1 f(y)\varphi_n^{(k)}(y)dy,$$

which holds for all values of i, k, and n.

In fact there occurs a phenomenon quite analogous to Gibbs's phenomenon for Fornier's series. For the set φ, the approximating functions are uniformly bounded. The peaks of the approximating function $S_n^{(k)}$ disappear entirely for $k = 2^{n-1}$ but reappear (usually altered in height) for for larger values of n.

It is clear that the facts concerning the approximating curves for $f(x)$ hold without essential modification for a function of limited variation at a simple finite discontinuity, and that the facts for the summation of the approximating sequence hold without essential modification for a function continuous except at a simple finite discontinuity.

§ 7. The Uniqueness of Expansions.

We now study the possibility of a series of the form

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots \tag{27}$$

which converges on $0 \leqq x \leqq 1$ to the sum zero, with the possible exception of a certain number of points x. Faber has pointed out* that there exists a series of the functions $\chi_n^{(k)}(x)$ which converges to zero except at one single point, and the convergence is uniform except in the neighborhood of that point.

We state for reference the easily proved

LEMMA. *If the series* (27) *converges for even one dyadically irrational value of x, then* $\lim_{n=\infty} a_n = 0$.

* L. c., p. 111.

This lemma results immediately from the fact that $\varphi_n^{(k)}(x) = \pm 1$ if x is dyadically irrational.*

We shall now use this lemma to establish

THEOREM IX. *If the series* (27) *converges to the sum zero uniformly except in the neighborhood of a single value of* x, *then* $a_n = 0$ *for every* n.

We phrase the argument to apply when this exceptional value x_1 is dyadically irrational. If $x_1 > \frac{1}{2}$, we have for $0 \leqq x \leqq \frac{1}{2}$,

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots = 0,$$
$$(a_0 + a_1)\varphi_0(y) + (a_2 + a_3)\varphi_1(y) + (a_4 + a_5)\varphi_2(y) + \cdots = 0,$$

for every value of $y = 2x$. Then we have from the uniformity of the convergence,

$$a_0 + a_1 = 0, \qquad a_2 + a_3 = 0, \qquad a_4 + a_5 = 0, \qquad \cdots. \tag{28}$$

If $x_1 < \frac{3}{4}$, we have for $\frac{3}{4} \leqq x \leqq 1$,

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots = 0,$$

or for $0 \leqq y \leqq 1$, $y = 4x - 3$,

$$(a_0 - a_1 + a_2 - a_3)\varphi_0(y) + (a_4 - a_5 + a_6 - a_7)\varphi_1(y) \\ + (a_{4n} - a_{4n+1} + a_{4n+2} - a_{4n+3})\varphi_n(y) + \cdots = 0.$$

From the uniformity of the convergence we have

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= 0, \\ a_4 - a_5 + a_6 - a_7 &= 0, \\ \cdots\cdots\cdots&\cdot, \end{aligned}$$

or from (28),

$$\begin{aligned} a_0 = -a_1 = -a_2 &= a_3, \\ a_4 = -a_5 = -a_6 &= a_7, \\ \cdots\cdots\cdots&\cdot. \end{aligned}$$

If $x_1 > \frac{5}{8}$, we have for $\frac{5}{8} \leqq x \leqq \frac{3}{4}$,

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots = 0,$$

or for $0 \leqq y \leqq 1$, $y = 8x - 5$,

$$(a_0 - a_1 - a_2 + a_3 - a_4 + a_5 + a_6 - a_7)\varphi_0(y) \\ + (a_8 - a_9 - a_{10} + a_{11} - a_{12} + a_{13} + a_4 - a_{15})\varphi_1(y) + \cdots = 0.$$

Then each of these coefficients must vanish, and hence

$$a_0 = -a_1 = -a_2 = a_3 = a_4 = -a_5 = -a_6 = a_7.$$

* This lemma is closely connected with a general theorem due to Osgood, *Transactions of the American Mathematical Society*, Vol. 10 (1909), pp. 337–346.
See also Plancherel, *Mathematische Annalen*, Vol. 68 (1909–1910), pp. 270–278.

Continuation in this way together with the Lemma shows that every a_n must vanish. This reasoning is typical and does not essentially depend on our numerical assumptions about x_1. Then Theorem IX is proved.

The reasoning is precisely similar if instead of the hypothesis of Theorem IX we admit the possibility of a finite number of points in the neighborhood of each of which the convergence is not assumed uniform:

THEOREM X. *If the series*

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots$$

converges to the sum zero uniformly, $0 \leqq x \leqq 1$, *except in the neighborhood of a finite number of points, then* $0 = a_1 = a_2 = \cdots = a_n = \cdots$.

HARVARD UNIVERSITY,
May, 1922.

CONGRUENCES DETERMINED BY A GIVEN SURFACE.

By Claribel Kendall.

§ 1. Introduction.

It has been shown by Professor Wilczynski* that a non-developable analytic surface S may be regarded as an integrating surface of a non-involutory, completely integrable system of partial differential equations of the form

$$(1)\qquad \begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned}$$

where the subscripts denote partial differentiation, and where the coefficients, which are seminvariants, are analytic functions of u and v satisfying the integrability conditions

$$(2)\qquad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0. \end{aligned}$$

Then the curves $u = \text{const.}$, $v = \text{const.}$ form an asymptotic net on the surface S. We shall assume that in general $a' \neq 0$, $b \neq 0$, thus excluding ruled surfaces from our discussion.†

Under the above conditions (1) has exactly four linearly independent solutions

$$(3)\qquad y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

which are interpreted as the homogeneous coördinates of a point y on the surface S. The semicovariants of (1) are‡

$$(4)\qquad y, \quad y_u, \quad y_v, \quad y_{uv}.$$

Substituting the values (3) for y in (4) we obtain four points y, y_u, y_v, y_{uv} which are not coplanar since no relations of the form

$$\alpha y^{(k)} + \beta y_u^{(k)} + \gamma y_v^{(k)} + \delta y_{uv}^{(k)} = 0 \qquad (k = 1, 2, 3, 4)$$

can exist among them. For, otherwise (1) could have at most three linearly

* "Projective Differential Geometry of Curved Surfaces," first memoir, *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 246–7.

† Loc. cit., p. 260.

‡ "Projective Differential Geometry of Curved Surfaces," second memoir, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 79. We shall hereafter refer to this paper as Second memoir.

independent solutions. Hence these points may be used as the vertices of a local tetrahedron of reference for the purpose of studying S in the neighborhood of the point y. An expression of the form

$$\tau = a_1 y + a_2 y_u + a_3 y_v + a_4 y_{uv},$$

where a_1, a_2, a_3, a_4 are analytic functions of u and v, assumes four values $\tau^{(1)}$, $\tau^{(2)}$, $\tau^{(3)}$, $\tau^{(4)}$ corresponding to the four values of y. Hence τ determines a point whose local coördinates may be defined by writing

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = a_3, \qquad x_4 = a_4.$$

Consider the case when two such points

$$\begin{aligned} \tau_1 &= a_1 y + a_2 y_u + a_3 y_v + a_4 y_{uv}, \\ \tau_2 &= b_1 y + b_2 y_u + b_3 y_v + b_4 y_{uv} \end{aligned} \tag{5}$$

are given for every point y of the surface S. If we associate the line l, determined by τ_1 and τ_2, with each point of S, these lines form a congruence. Wilczynski and Green have considered such congruences in cases where the lines l pass through the point y or lie in the corresponding tangent plane. In this paper we shall consider the more general problems connected with the congruences determined by the lines l when l has an arbitrary position relative to S. General formulas will be obtained for the torsal curves and the guide curves (to be defined later), and for the focal points on l. These general formulas will then be applied to certain special congruences in connection with which various configurations of the lines themselves will be studied.

§ 2. Determination of the Developables of a Congruence.

For each point y of the surface S the points (5) determine a line l whose homogeneous line coördinates are

$$\omega_{ik} = a_i b_k - a_k b_i \qquad (i, k = 1, 2, 3, 4).$$

We wish to find the curves on S along which y must move in order that the corresponding line l of the congruence may describe a developable. These curves will be called the torsal curves of the surface S with respect to the congruence. Let u and v increase by amounts du and dv, where du and dv are infinitesimals, in such a way that the point y will change to $y + dy$, a new point on one of the torsal curves. Then the points τ_1 and τ_2 will move to $\tau_1 + d\tau_1$ and $\tau_2 + d\tau_2$ respectively. The line joining $\tau_1 + d\tau_1$ to $\tau_2 + d\tau_2$ is a generator of the developable consecutive to $\tau_1\tau_2$ and must intersect $\tau_1\tau_2$. Therefore τ_1, τ_2, $d\tau_1$, $d\tau_2$ must be coplanar. Now

$$d\tau_1 = (\tau_1)_u du + (\tau_1)_v dv, \qquad d\tau_2 = (\tau_2)_u du + (\tau_2)_v dv.$$

Hence, using (5) and (1),

$$
(6)\quad
\begin{aligned}
d\tau_1 &= (A_1du + A_1'dv)y + (A_2du + A_2'dv)y_u \\
&\qquad + (A_3du + A_3'dv)y_v + (A_4du + A_4'dv)y_{uv}, \\
d\tau_2 &= (B_1du + B_1'dv)y + (B_2du + B_2'dv)y_u \\
&\qquad + (B_3du + B_3'dv)y_v + (B_4du + B_4'dv)y_{uv},
\end{aligned}
$$

where

$$
(7)\quad
\begin{aligned}
A_1 &= (a_1)_u - fa_2 + (2bg - f_v)a_4, & A_1' &= (a_1)_v - ga_3 + (2a'f - g_u)a_4, \\
A_2 &= (a_2)_u + a_1 + 4a'ba_4, & A_2' &= (a_2)_v - 2a'a_3 - (g + 2a_u')a_4, \\
A_3 &= (a_3)_u - 2ba_2 - (f + 2b_v)a_4, & A_3' &= (a_3)_v + a_1 + 4a'ba_4, \\
A_4 &= (a_4)_u + a_3, & A_4' &= (a_4)_v + a_2, \\
B_1 &= (b_1)_u - fb_2 + (2bg - f_v)b_4, & B_1' &= (b_1)_v - gb_3 + (2a'f - g_u)b_4, \\
B_2 &= (b_2)_u + b_1 + 4a'bb_4, & B_2' &= (b_2)_v - 2a'b_3 - (g + 2a_u')b_4, \\
B_3 &= (b_3)_u - 2bb_2 - (f + 2b_v)b_4, & B_3' &= (b_3)_v + b_1 + 4a'bb_4, \\
B_4 &= (b_4)_u + b_3, & B_4' &= (b_4)_v + b_2.
\end{aligned}
$$

A necessary and sufficient condition that the points τ_1, τ_2, $d\tau_1$, $d\tau_2$ lie in a plane is that the determinant of the coördinates of the four points be zero. Expanding the determinant we obtain

$$
(8)\qquad Ldu^2 + 2Mdudv + Ndv^2 = 0,
$$

where

$$
\begin{aligned}
L &= \omega_{12}(A_3B_4) + \omega_{13}(A_4B_2) + \omega_{14}(A_2B_3) \\
&\qquad + \omega_{23}(A_1B_4) + \omega_{42}(A_1B_3) + \omega_{34}(A_1B_2), \\
2M &= \omega_{12}[(A_3B_4') + (A_3'B_4)] + \omega_{13}[(A_4B_2') + (A_4'B_2)] \\
&\qquad + \omega_{14}[(A_2B_3') + (A_2'B_3)] + \omega_{23}[(A_1B_4') + (A_1'B_4)] \\
&\qquad + \omega_{42}[(A_1B_3') + (A_1'B_3)] + \omega_{34}[(A_1B_2') + (A_1'B_2)], \\
N &= \omega_{12}(A_3'B_4') + \omega_{13}(A_4'B_2') + \omega_{14}(A_2'B_3') \\
&\qquad + \omega_{23}(A_1'B_4') + \omega_{42}(A_1'B_3') + \omega_{34}(A_1'B_2').
\end{aligned}
$$

Here (A_3B_4), etc., are the determinantal expressions $A_3B_4 - A_4B_3$, etc. We may then conclude

The torsal curves of the surface with respect to the congruence are determined by (8), *a quadratic differential equation which determines a net of curves on the surface S.*

§ 3. The Focal Points of the Lines *l*.

Each line l of the congruence belongs to two developables of the congruence, and the two points in which l touches the cuspidal edges of these developables (viz., the focal points of the line l) will now be found. Any point on the line l is given by an expression of the form

$$
(9)\qquad \varphi = \lambda\tau_1 + \mu\tau_2,
$$

where τ_1 and τ_2 are given by (5) and λ and μ are arbitrary functions of u and v. If φ is to be a focal point of l, then the tangent plane to the surface formed by all the points φ, as u and v vary, must contain the line l. Hence τ_1, τ_2, φ_u, φ_v must be coplanar. Noting that

$$\varphi_u = \lambda_u\tau_1 + \mu_u\tau_2 + \lambda(\tau_1)_u + \mu(\tau_2)_u,$$
$$\varphi_v = \lambda_v\tau_1 + \mu_v\tau_2 + \lambda(\tau_1)_v + \mu(\tau_2)_v,$$

it follows that since τ_1 and τ_2 are coplanar with $\varphi_u - \lambda_u\tau_1 - \mu_u\tau_2$ and $\varphi_v - \lambda_v\tau_1 - \mu_v\tau_2$ they are also coplanar with $\lambda(\tau_1)_u + \mu(\tau_2)_u$ and $\lambda(\tau_1)_v + \mu(\tau_2)_v$. So that a necessary and sufficient condition that these points lie in the same plane is that the determinant of the coördinates of the points τ_1, τ_2, $\lambda(\tau_1)_u + \mu(\tau_2)_u$, $\lambda(\tau_1)_v + \mu(\tau_2)_v$ be zero. Expanding this determinant and using (7) we obtain

$$L'\lambda^2 + 2M'\lambda\mu + N'\mu^2 = 0, \tag{10}$$

where

$$\begin{aligned}
L' &= \omega_{12}(A_3A_4') + \omega_{13}(A_4A_2') + \omega_{14}(A_2A_3') \\
&\qquad + \omega_{23}(A_1A_4') + \omega_{42}(A_1A_3') + \omega_{34}(A_1A_2'), \\
2M' &= \omega_{12}[(A_3B_4') + (B_3A_4')] + \omega_{13}[(A_4B_2') + (B_4A_2')] \\
&\qquad + \omega_{14}[(A_2B_3') + (B_2A_3')] + \omega_{23}[(A_1B_4') + (B_1A_4')] \\
&\qquad + \omega_{42}[(A_1B_3') + (B_1A_3')] + \omega_{34}[(A_1B_2') + (B_1A_2')], \\
N' &= \omega_{12}(B_3B_4') + \omega_{13}(B_4B_2') + \omega_{14}(B_2B_3') \\
&\qquad + \omega_{23}(B_1B_4') + \omega_{42}(B_1B_3') + \omega_{34}(B_1B_2').
\end{aligned}$$

Here we have determinantal quantities similar to those in (8). The two values of λ/μ obtained from (10), substituted in (9), give the focal points of the line l, viz.,

$$\varphi_1 = \lambda_1\tau_1 + \mu_1\tau_2, \qquad \varphi_2 = \lambda_2\tau_1 + \mu_2\tau_2. \tag{11}$$

Their product determines a covariant

$$N'\tau_1^2 - 2M'\tau_1\tau_2 + L'\tau_2^2. \tag{12}$$

We may then conclude

The focal points of the lines l of the congruence are given by the factors of the covariant expression (12).

§ 4. The Guide Curves of the Congruence.

With each point y of the surface S is associated a unique plane through it, viz., the plane containing the line l determined by τ_1 and τ_2. This plane intersects the tangent plane at y in a well-determined line unless l lies in the tangent plane or passes through y. We shall now find the family of curves on S which will have these lines as tangents. These curves will

be called the *guide curves* of the surface with respect to the given congruence since the line l acts as a guide in determining the position of the plane. The equation of this plane is

$$\omega_{34}x_2 + \omega_{42}x_3 + \omega_{23}x_4 = 0. \tag{13}$$

It intersects the tangent plane $x_4 = 0$ in the line

$$\omega_{34}x_2 + \omega_{42}x_3 = 0, \qquad x_4 = 0.$$

If $u = u(t)$, $v = v(t)$ is the equation of the desired curve through y, the condition that this line be tangent to the curve gives

$$\omega_{34}du + \omega_{42}dv = 0 \tag{14}$$

as the differential equation of the required curves on S.

Miss Sperry* has developed the differential equation for the union curves on the surface S, which are curves such that the osculating plane of a point y on the curve will contain the generator of a congruence when this generator passes through the point y and does not lie in the tangent plane to S at y. We now desire to find the condition under which the guide curves are also union curves.

As before let the equation of a curve on S be given by $u = u(t)$, $v = v(t)$. The osculating plane of this curve at one of its points y is

$$2u'v'^2x_2 - 2u'^2v'x_3 - (u''v' - u'v'' + 2bu'^3 - 2a'v'^3)x_4 = 0, \tag{15}$$

where accents indicate differentiation as to t. If (13) and (15) are to represent the same plane,

$$\frac{2u'v'^2}{\omega_{34}} = \frac{-2u'^2v'}{\omega_{42}} = \frac{u'v'' - v'u'' - 2bu'^3 + 2a'v'^3}{\omega_{23}}. \tag{16}$$

$u' = 0$, $v' = 0$ is a solution of these equations but in this case the coordinates of the point y' would be (0, 0, 0, 0) which is not admissible. Assuming $u' \neq 0$ and introducing u as the parameter instead of t we obtain

$$\frac{dv}{du} = \frac{\omega_{34}}{\omega_{42}}, \qquad 2\omega_{23}\left(\frac{dv}{du}\right)^2 = \omega_{34}\left[\frac{d^2v}{du^2} - 2b + 2a'\left(\frac{dv}{du}\right)^3\right]. \tag{17}$$

Substituting the value of dv/du from the first equation of (17), which is the same as (14), into the second we find

$$\begin{aligned} 2\omega_{23}\omega_{34}\omega_{42} - \omega_{34}\omega_{42}(\omega_{42})_u + \omega_{42}^2(\omega_{34})_u \\ + \omega_{34}^2(\omega_{42})_v - \omega_{42}\omega_{34}(\omega_{34})_v + 2b\omega_{42}^3 + 2a'\omega_{44}^3 = 0 \end{aligned} \tag{18}$$

* "Properties of a Certain Projectively Defined Two-parameter Family of Curves on a General Surface," *American Journal of Mathematics*, Vol. XL (1918), pp. 213–224.

as the relation that must be satisfied by the line coördinates of l in order that the guide curves may be union curves.

Condition (18) is satisfied by $\omega_{34} = \omega_{42} = 0$. Two cases may arise depending on whether $\omega_{23} = 0$ or $\omega_{23} \neq 0$. If $\omega_{23} = 0$, we see from (16) that the solution is $u' = 0, v' = 0$, which has been excluded. Geometrically this occurs when the line l passes through the point y. As stated above this case has been considered by Miss Sperry from another point of view. If $\omega_{23} \neq 0$, the solution of (16) is $u' = 0$ if $v' \neq 0$ and is $v' = 0$ if $u' \neq 0$. This is the case when the line l lies in the tangent plane to S at y and does not pass through y.

We may then conclude

The one-parameter family of guide curves of the surface with respect to the given congruence has (14) *for its differential equation. In case* (18) *is satisfied the guide curves are also union curves. When l passes through the point y on the surface S or lies in the tangent plane to S at y, the guide curves are indeterminate.*

§ 5. The Osc-scroll-flec Congruences.

We shall now apply these results to some special congruences closely associated with a given surface. We begin by recalling what Wilczynski calls the osculating ruled surfaces of the first and second kinds, respectively. One of these, R_1, is the locus of the tangents to the asymptotic curves $v =$ const. along a fixed curve $u =$ const., and R_2 is the locus of the tangents to the asymptotic curves $u =$ const. along a fixed curve $v =$ const. The differential equations of R_1 and R_2 referred to our local tetrahedron of reference are given in Wilczynski's second memoir, pp. 81–82.

We shall have occasion to use the invariants of weights four, nine, and ten for R_1 and R_2. Expressed in terms of the coefficients of (1), for R_1 they are*

$$
\begin{aligned}
\theta_4 &= 2^6(a_u'^2 - 2a'a_{uu}' - 4a'^2 f - 4a'^2 b_v), \\
\theta_9 &= 4(C^3 + 8a'CC_{uv} - 8a'C_uC_v), \qquad (19)\\
\theta_{10} &= 2^6 a'^2 C_u^2 - C^2\theta_4,
\end{aligned}
$$

where†

$$
C = 8a_{uv}' - 8\frac{a_u'a_v'}{a'} - 32a'^2 b. \qquad (20)
$$

In obtaining θ_{10} and θ_9 use has been made of the relation

$$
2a_v'\theta_4 - a'(\theta_4)_v = 16a'^2 C_u. \qquad (21)
$$

The corresponding invariants θ_4', θ_9', θ_{10}' for R_2 are obtained from (19) by

* Second memoir, pp. 81, 84. θ_9 and θ_{10} were obtained by C. D. Meacham, a student at the University of Chicago.

† Loc. cit., p. 84.

the transpositions

$$(a'b), \quad (fg), \quad (uv), \quad (\theta_4\theta_4'), \quad (CC'), \tag{22}$$

where C' is obtained from C by the same transpositions.

The vanishing of θ_4 is the condition that R_1 have coincident branches to its flecnode curve, and $\theta_4' = 0$ is the corresponding condition for R_2. Consider for the present the case where $\theta_4 \neq 0$. Then R_1 may be referred to its flecnode curve. Let us denote the flecnodes on the generator yy_u of R_1 by η and ζ and let the lines ηr and ζs* be the flecnode tangents of R_1 at η and ζ respectively. To every point y of S there belongs such a line ηr and hence we obtain a congruence associated with S. Similarly ζs generates a second congruence. Relative to the osculating ruled surface R_2 two other congruences are determined in this way provided that the branches of the flecnode curve of R_2 do not coincide, i.e., provided $\theta_4' \neq 0$. We shall call these four congruences the *osc-scroll-flec congruences,* and we shall designate the congruences determined by ηr and ζs by Γ_1 and Γ_1' respectively and the corresponding congruences associated with R_2 by Γ_2 and Γ_2'. Referred to our local tetrahedron of reference, η, r, ζ, s are given by†

$$\begin{aligned}
-32a'\sqrt{\theta_4}\,\eta &= (8a_u' - \sqrt{\theta_4})y - 16a'y_u, \\
-32a'\sqrt{\theta_4}\,r &= 64a'^2by + 2(8a_u' - \sqrt{\theta_4})y_v - 32a'y_{uv}, \\
32a'\sqrt{\theta_4}\,\zeta &= (8a_u' + \sqrt{\theta_4})y - 16a'y_u, \\
32a'\sqrt{\theta_4}\,s &= 64a'^2by + 2(8a_u' + \sqrt{\theta_4})y_v - 32a'y_{uv}.
\end{aligned} \tag{23}$$

Hence, for the congruence Γ_1, the two points τ_1 and τ_2 of our general theory are given by

$$\begin{aligned}
\tau_1 &= (8a_u' - \sqrt{\theta_4})y - 16a'y_u, \\
\tau_2 &= 64a'^2by + 2(8a_u' - \sqrt{\theta_4})y_v - 32a'y_{uv}.
\end{aligned} \tag{24}$$

The corresponding formulas for the congruence Γ_1' differ from (24) merely in the sign of the radical.

Substituting (24) in (8) of § 2 and making use of (2), (19), (20), (21) we find that the torsal curves for Γ_1 are given by

$$\begin{aligned}
[(\theta_4)_u^2 - 16a'^2\theta_4\theta_4']du^2 - 4(\theta_4)_u(8a'C_u + C\sqrt{\theta_4})du\,dv \\
+ 4(8a'C_u + C\sqrt{\theta_4})^2dv^2 = 0.
\end{aligned} \tag{25}$$

By changing the sign of the radical in (25) we obtain the torsal curves for

* Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces." Teubner, Leipzig, 1906, p. 124. We shall hereafter refer to this work as "Proj. Diff. Geom." The quantities r and s are there referred to as ρ and σ.

† Second memoir, p. 84.

Γ_1', viz.,

$$(26)\qquad \begin{aligned}[(\theta_4)_u^2 - 16a'^2\theta_4\theta_4']du^2 - 4(\theta_4)_u(8a'C_u - C\sqrt{\theta_4})dudv \\ + 4(8a'C_u - C\sqrt{\theta_4})^2dv^2 = 0.\end{aligned}$$

The torsal curves for Γ_2 and Γ_2' may be obtained from (25) and (26) by means of the transpositions (22).

When $\theta_4 = 0$, R_1 has but one branch to its flecnode curve. In that case the single flecnode tangent may be determined by*

$$\begin{aligned}2a'\zeta &= a_u'y - 2a'y_u, \\ 2a's &= 8a'^2by + 2a_u'y_v - 4a'y_{uv},\end{aligned}$$

where ζ is the flecnode. Γ_1 and Γ_1' coincide and the torsal curves for this congruence are found to be

$$(27)\qquad 4a'^2\theta_4'du^2 - C^2dv^2 = 0,$$

a conjugate net on S.

The focal points of the osc-scroll-flec congruences are found by factoring the covariant expression (12) of § 3. For Γ_1 when $\theta_4 \neq 0$, this covariant is

$$(28)\qquad (64b_v^2 - \theta_4')\tau_1^2 - 128bb_v\tau_1\tau_2 + 64b^2\tau_2^2,$$

aside from a factor $8a'C_u + C\sqrt{\theta_4}$ which is, in general, different from zero. Its vanishing makes $\theta_{10} = 0$. But $\theta_4 \neq 0$, $\theta_{10} = 0$ is the condition that R_1 have a straight line directrix.† An examination of the equations of the ruled surface R_1 when referred to its flecnode curves‡ shows that the flecnode curve C_η is also an asymptotic curve on R_1 and consequently is a straight line,§ the straight line directrix of R_1. Equation (25) shows that in this case the torsal net reduces to $du^2 = 0$. As y moves along $u =$ const., R_1 remains the same, the flecnode curve is a straight line and is, in fact, the line ηr; hence we see that the developable generated by ηr reduces to a straight line and consequently the congruence Γ_1 degenerates into a ruled surface.

For the congruence Γ_1' we obtain the same expression (28) for the focal points, but with the factor $8a'C_u - C\sqrt{\theta_4}$ omitted instead of $8a'C_u + C\sqrt{\theta_4}$. Its vanishing would cause θ_{10} to vanish and Γ_1' would degenerate into a ruled surface. If both of the factors of θ_{10} vanish, R_1 will have two straight line directrices and hence will belong to a linear congruence, and Γ_1 and Γ_1' will degenerate into ruled surfaces.

* Second memoir, p. 86.
† "Proj. Diff. Geom.," p. 167.
‡ Second memoir, pp. 83–84.
§ "Proj. Diff. Geom.," p. 150.

When $\theta_4 = 0$, the congruences Γ_1, Γ_1' coincide. The focal points are given by (28) aside from a factor C which is, in general, non-vanishing. If $C = 0$, the congruence degenerates into a ruled surface.

The focal points for the congruences Γ_2 and Γ_2', whether distinct or coincident, are obtained from (28) by means of the transpositions (22).

From the theorem of § 4 we find that the guide curves are given by $v =$ const. for Γ_1 and Γ_1', and by $u =$ const. for Γ_2 and Γ_2', as is obvious geometrically. The quantity $\omega_{34} = 0$ while $\omega_{42} \neq 0$, for Γ_1 and Γ_1', and $\omega_{42} = 0$, $\omega_{34} \neq 0$ for Γ_2 and Γ_2', hence the condition that the guide curves be union curves is not satisfied and the union curves do not exist.

In order to interpret geometrically special cases arising out of the discussion of the equations which have just been found, it will be convenient to give some geometric properties which follow when certain of the invariants are zero.

As previously mentioned, $\theta_4 = 0$ is the condition that R_1 may have coincident branches to its flecnode curve. It is also the condition that the focal sheets of the osc-scroll-flec congruences associated with R_2 coincide, as may be seen from the equation obtained from (28) by (22). In this case the cuspidal edges on this focal sheet are asymptotic curves on that surface. We also noted that $\theta_4 \neq 0$, $\theta_{10} = 0$ is the condition for R_1 to have a straight line directrix. Corresponding conditions hold when $\theta_4' = 0$ and when $\theta_4' \neq 0$, $\theta_{10}' = 0$. The conditions $C = 0$ and $C' = 0$ correspond to the cases when the asymptotic curves $v =$ const. and $u =$ const., respectively, belong to linear complexes.* If $C = C' = 0$, $\theta_4 \neq 0$, $\theta_4' \neq 0$, R_1 and R_2 belong to linear congruences with distinct directrices. If $C = \theta_4 = 0$, $C' = \theta_4' = 0$, these linear congruences have coincident directrices.†

By an examination of the discriminant of (25) for $\theta_4 \neq 0$, and of (27) for $\theta_4 = 0$, we find that the torsal curves for Γ_1 represent a one-parameter family of curves instead of a proper net in a number of special cases, which may be interpreted geometrically in the light of the foregoing properties. Among these special cases we find several where the asymptotic curves on S are torsal curves. When $u =$ const. is a torsal curve, Γ_1 degenerates into a ruled surface. When $v =$ const. is a torsal curve, the focal sheets of Γ_1 coincide. An examination of (25) shows that the asymptotic curves of S cannot both be torsal curves under the same conditions. Furthermore, the torsal curves form conjugate nets under special conditions which may be interpreted geometrically. Corresponding conditions hold for the torsal curves of Γ_1' and we can readily find the conditions under which these

* C. T. Sullivan, "Properties of Surfaces whose Asymptotic Curves belong to Linear Complexes," *Transactions of the American Mathematical Society*, Vol. 15 (1914), p. 178.

† Second memoir, p. 86.

curves coincide with the torsal curves of Γ'_1. Similar conditions hold relative to the torsal curves of the Γ_2 and Γ'_2 congruences.

We shall now return to some general considerations regarding the osc-scroll-flec congruences. The tangents to the two torsal curves of Γ_1 at a point y of the surface are the lines joining y to the two points

$$y_u + (dv/du)_1 y_v, \qquad y_u + (dv/du)_2 y_v,$$

where $(dv/du)_1$ and $(dv/du)_2$ are the two roots of equation (25) regarded as a quadratic in dv/du. Let t_1 and t_2 be these two tangents. The directions conjugate to t_1 and t_2 are obtained by joining the point y to the points

$$y_u - (dv/du)_1 y_v, \qquad y_u - (dv/du)_2 y_v,$$

respectively. Using the term employed by Green* we shall call these two new tangents the reflected tangents of t_1 and t_2. The totality of these reflected tangents determines a new net on S which may, with Green, be called the reflected Γ_1-curves. They are determined by the differential equation

$$\begin{aligned}(29)\quad [(\theta_4)_u^2 - 16a'^2\theta_4\theta'_4]du^2 + 4(\theta_4)_u(8a'C_u + C\sqrt{\theta_4})du\,dv \\ + 4(8a'C_u + C\sqrt{\theta_4})^2 dv^2 = 0,\end{aligned}$$

an equation differing from (25) in the sign of the middle term only, since its roots are those of (25) with the signs changed. The torsal curves of Γ_1, the reflected Γ_1-curves, and the asymptotic curves at any point y of the surface S thus constitute three pairs in an involution. The Jacobian of the torsal curves of Γ_1 and the reflected Γ_1-curves gives the double elements of the involution, which constitute, of course, a pair of conjugate tangents, namely

$$(30)\qquad [(\theta_4)_u^2 - 16a'^2\theta_4\theta'_4]du^2 - 4(8a'C_u + C\sqrt{\theta_4})^2 dv^2 = 0.$$

We may then say

The torsal curves of each osc-scroll-flec congruence, the corresponding reflected curves, and the asymptotic curves at any point y of the surface S constitute three pairs in an involution. The double elements of the involutions so determined give four unique projectively defined conjugate nets on S, one relative to each of the osc-scroll-flec congruences. Their differential equations are given by (30) *and the equations obtained from* (30) *by changing the sign of the radical and by the transpositions* (22) *applied to these two equations.*

The focal points of the line ηr for the Γ_1-congruence were found to be given by the expressions

$$\varphi_1 = \lambda_1\tau_1 + \mu_1\tau_2, \qquad \varphi_2 = \lambda_2\tau_1 + \mu_2\tau_2,$$

* Memoir on the general theory of surfaces and rectilinear congruences. *Transactions of the American Mathematical Society*, Vol. 20 (1919), p. 93.

where λ_1/μ_1 and λ_2/μ_2 are the roots of the quadratic equation

$$64b^2\lambda^2 + 128bb_v\lambda\mu + (64b_v^2 - \theta_4')\mu^2 = 0,$$

provided $\theta_{10} \neq 0$. A point which proves to be of considerable interest is obtained by finding the harmonic conjugate of η, i.e., τ_1, with respect to φ_1 and φ_2. It is given by

$$\alpha = 1/2(\lambda_1/\mu_1 + \lambda_2/\mu_2)\tau_1 + \tau_2.$$

From the above quadratic equation in λ/μ and from (24) this is found to be the point

$$\begin{aligned}(31)\qquad \alpha = (-\,8a_u'b_v + 64a'^2b^2 + b_v\sqrt{\theta_4})y + 16a'b_vy_u \\ + 2b(8a_u' - \sqrt{\theta_4})y_v - 32a'by_{uv}.\end{aligned}$$

The corresponding point on ζs of the congruence Γ_1' is given by

$$\begin{aligned}(32)\qquad \beta = (-\,8a_u'b_v + 64a'^2b^2 - b_v\sqrt{\theta_4})y + 16a'b_vy_u \\ + 2b(8a_u' + \sqrt{\theta_4})y_v - 32a'by_{uv}.\end{aligned}$$

Relative to Γ_2 and Γ_2' we obtain the two points α' and β', found from (31) and (32) by the transpositions (22).

The equation of the osculating quadric Q of the surface S at the point y is*

$$(33)\qquad x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0.$$

It can readily be shown that *the lines $\alpha\beta$ and $\alpha'\beta'$ lie on this quadric and that they intersect one another at the point where the directrix of the second kind d'*† *intersects the osculating quadric.* $\alpha\beta$ intersects the side yy_u of the local tetrahedron of reference and $\alpha'\beta'$ intersects the side yy_v.

For each point y on S a line $\alpha\beta$ is determined even when $\theta_4 = 0$. If $\theta_4 = 0$, α and β coincide but the line $\alpha\beta$ is then the line joining $\alpha = \beta$ to the point where d' intersects the osculating quadric. Similarly for $\alpha'\beta'$. Consequently we have two new congruences associated with S. The torsal curves for these congruences are given by

$$\begin{aligned}(34)\qquad &2^8b^2C'^2du^2 - 2^5b\theta_4'(C' + 32a'b^2)dudv + (\theta_4'^2 - 2^{10}b^4\theta_4)dv^2 = 0,\\ &(\theta_4^2 - 2^{10}a'^4\theta_4')du^2 - 2^5a'\theta_4(C + 32a'^2b)dudv + 2^8a'^2C^2dv^2 = 0,\end{aligned}$$

respectively. If $\theta_4' = 0$, the torsal curves given in the first equation of (34) determine a conjugate net which coincides with the torsal net for the coincident Γ_2 and Γ_2' congruences. In this case the focal sheets of Γ_1 and Γ_1' coincide and $\alpha\beta$ becomes the line joining the coincident focal points on ηr and ζs. The line $\alpha'\beta'$ is now the line joining $\alpha' = \beta'$ to the point where d'

* Second memoir, p. 82.

† Loc. cit., p. 97.

intersects the osculating quadric. Similar conditions hold if $\theta_4 = 0$. If these conditions occur simultaneously, i.e., if $\theta_4' = \theta_4 = 0$, the torsal curves of the two congruences reduce to $du^2 = 0$ and to $dv^2 = 0$, respectively.

The focal points on $\alpha\beta$ and $\alpha'\beta'$ are given by the factors of the covariant expressions

$$
(35) \qquad \begin{aligned} &2^{10}b^2C'\beta^2 - 2^8a'b^2\theta_4'\alpha\beta - (a'\theta_4'^2 + 16b^2C'\theta_4)\alpha^2, \\ &2^{10}a'^2C\beta'^2 - 2^8a'^2b\theta_4\alpha'\beta' - (b\theta_4^2 + 16a'^2C\theta_4')\alpha'^2, \end{aligned}
$$

respectively. For $\theta_4' = 0$, the focal points on $\alpha\beta$ separate α and β harmonically. Similarly in the case $\theta_4 = 0$, α' and β' are separated harmonically by the focal points of the line $\alpha'\beta'$. When $\theta_4 = \theta_4' = 0$, the focal points of $\alpha\beta$ are coincident with β and the focal points of $\alpha'\beta'$ are coincident with β', i.e., the focal sheets of both of these congruences coincide.

The guide curves for $\alpha\beta$ and $\alpha'\beta'$ are found to be $u =$ const. and v = const., respectively, as is obvious geometrically.

§ 6. The Congruences Determined by the Pairs of Complexes C_1, C' and C_2, C''.

Associated with a point y of the surface S there are four complexes,— the two complexes C_1 and C_2 which osculate the ruled surfaces R_1 and R_2 respectively and the two complexes C' and C'' which osculate the asymptotic curves of the first and second kinds respectively. Four of the pairs of complexes obtained from these are in involution, i.e., their bilinear invariants are zero.* Professor Wilczynski has considered† more in detail the congruences obtained from the directrices of the congruence common to the osculating complexes C' and C''. These he called the directrix congruences of the first and second kinds. We shall proceed to consider the other three pairs which are in involution. They are the pairs C_1, C'; C_2, C''; and C_1, C_2. The first two pairs may be considered together since they are symmetrically situated with respect to the surface S.

The equations of the complexes C_1 and C_2 referred to our local tetrahedron of reference are given by‡

$$
(36) \qquad C_1 : a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{34}\omega_{34} + a_{42}\omega_{42} = 0,
$$

where

$$
\begin{aligned} a_{12} &= 0, \quad a_{13} = 2^8a'^2C, \quad a_{14} = a_{23} = 2^7a'(a'C_u + a_u'C), \\ a_{34} &= -2^9a'^3bC, \quad a_{42} = -[C(\theta_4 + 64a_u'^2) + 2^7a'a_u'C_u], \end{aligned}
$$

* Second memoir, p. 95.

† Loc. cit., pp. 114–120.

‡ Loc. cit., pp. 85, 86, 89.

with the invariant

$$(37)\qquad A = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 2^8 a'^2 \theta_{10};$$

and

$$(38)\qquad C_2 : b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{34}\omega_{34} + b_{42}\omega_{42} = 0,$$

where

$$b_{12} = 2^8 b^2 C', \qquad b_{13} = 0, \qquad b_{14} = -b_{23} = 2^7 b(bC'_v + b_v C'),$$
$$b_{34} = C'(\theta'_4 + 64b_v^2) + 2^7 b b_v C'_v, \qquad b_{42} = 2^9 a' b^3 C',$$

with the invariant

$$(39)\qquad B = b_{12}b_{34} + b_{13}b_{42} + b_{14}b_{23} = -2^8 b^2 \theta'_{10}.$$

In writing down the above coefficients and invariants use has been made of (19) and (21). The equations of the complexes C' and C'' are*

$$(40)\qquad C' : -b_v\omega_{34} - b\omega_{14} + b\omega_{23} = 0,$$

with the invariant $A' = -b^2$,

$$(41)\qquad C'' : -a'_u\omega_{42} + a'\omega_{14} + a'\omega_{23} = 0,$$

with the invariant $A'' = a'^2$. It is to be noted that (38), (39) and (41) follow from (36), (37) and (40) respectively by applying the transpositions (22) and by interchanging the subscripts 2 and 3.

Consider the congruence determined by the linear complexes C_1 and C'. It will have two directrices, d_1 and d'_1, say. For every surface point we have two lines so determined. Hence d_1 and d'_1 will determine congruences associated with the surface S. We shall speak of these as the d_1-congruence and the d'_1-congruence. The equations of these directrices referred to the local tetrahedron of reference are found to be†

$$(42)\qquad \begin{aligned} a_{13}x_3 + (a_{14} + 16a'\sqrt{\theta_{10}})x_4 &= 0, \\ ba_{13}x_1 + b(a_{14} - 16a'\sqrt{\theta_{10}})x_2 - (ba_{34} + 16a'b_v\sqrt{\theta_{10}})x_4 &= 0, \end{aligned}$$

for d_1 and

$$(43)\qquad \begin{aligned} a_{13}x_3 + (a_{14} - 16a'\sqrt{\theta_{10}})x_4 &= 0, \\ ba_{13}x_1 + b(a_{14} + 16a'\sqrt{\theta_{10}})x_2 - (ba_{34} - 16a'b_v\sqrt{\theta_{10}})x_4 &= 0, \end{aligned}$$

for d'_1. In finding (42) and (43) it is useful to note that

$$a_{13}a_{24} - a_{14}^2 = 2^8 a'^2 \theta_{10}.$$

---

* Second memoir, pp. 92, 94.

† In obtaining these equations use has been made of equations (67)–(69) on pp. 94–95 of Second memoir where (69) should read $A''\omega^2 - (A', A'')\omega + A' = 0$.

Since equations (43) differ from (42) only in the sign of the radical, we shall discuss the d_1-congruence in detail and obtain results for the other by changing the sign of $\sqrt{\theta_{10}}$.

As the two points (5) determining d_1 we may take the points of intersection of d_1 with the planes $x_2 = 0$ and $x_3 = 0$. d_1 intersects the edge yy_u of the local tetrahedron of reference, as must obviously be the case since yy_u is a line of both C_1 and C'. After substituting from (36) we have

$$\begin{aligned} &\tau_1 = (8a'C_u + 8a'_uC - \sqrt{\theta_{10}})y - 16a'Cy_u, \\ (44)\quad &\tau_2 = (32a'^2b^2C - b_v\sqrt{\theta_{10}})y \\ &\qquad + (8a'bC_u + 8ba'_uC + b\sqrt{\theta_{10}})y_v - 16a'bCy_{uv}. \end{aligned}$$

Let τ_1 and σ_1 be the points in which d_1 intersects the osculating quadric Q whose equation is given in (33). τ_1 is given in (44) and

$$(45)\quad \begin{aligned} \sigma_1 = [8(a'b_vC_u + a'_ub_vC - 8a'^2b^2C) + b_v\sqrt{\theta_{10}}]y - 16a'b_vCy_u \\ - 2b(8a'C_u + 8a'_uC + \sqrt{\theta_{10}})y_v + 32a'bCy_{uv}. \end{aligned}$$

Let τ'_1 and σ'_1 be the points in which d'_1 intersects Q. The coördinates of τ'_1 and σ'_1 are obtained from those of τ_1 and σ_1 respectively by changing the sign of $\sqrt{\theta_{10}}$. It can readily be shown that *the lines $\sigma_1\sigma'_1$, $\tau_1\sigma'_1$, and $\tau'_1\sigma_1$ lie on the quadric.* Moreover, *the line $\sigma_1\sigma'_1$ coincides with the line $\alpha\beta$ of* § 5. It can easily be verified that τ_1 *and* τ'_1 *are the complex points* on yy_u and hence are harmonically separated by η and ζ, the flecnodes on yy_u.* Then since ηr, ζs, $\tau_1\sigma'_1$ and $\tau'_1\sigma_1$ are all generators of the same kind on the osculating quadric, σ_1 *and* σ'_1 *are harmonically separated by α and β.* In § 5 we saw that $\alpha\beta$, i.e., $\sigma_1\sigma'_1$, *passes through the point where the directrix d' of the second kind intersects the osculating quadric.* Similar conditions hold relative to the lines determined in the same way from the complexes C_2, C''.

The torsal curves for the d_1-congruence are found, except for a non-vanishing factor $C/a'\theta_4\theta_{10}$, to be

$$(46)\qquad L_1du^2 + 2M_1dudv + N_1dv^2 = 0,$$

where

$$\begin{aligned} L_1 &= b^2C[C(\theta_4P + 32a'^2(\theta_4)_uC_u)^2 + 16a'^2C\theta_{10}(16a'^2\theta_4\theta'_4 - (\theta_4)^2_u) \\ &\qquad - 2^8a'^3C'\theta_4\theta_{10}\sqrt{\theta_{10}}], \\ 2M_1 &= 4a'b\theta_4[4a'C'\theta^2_{10} - bC\theta_9(\theta_4P + 32a'^2(\theta_4)_uC_u)], \\ N_1 &= a'^2\theta_4[4b^2\theta_4\theta^2_9 - \theta^2_{10}\theta'_4 - 16b^2C\theta_4\theta_{10}\sqrt{\theta_{10}}], \end{aligned}$$

in which

$$P = C\theta_4 - 2^6a'^2C_{uu} - 2^6a'a'_uC_u.$$

* "Proj. Diff. Geom.," p. 208.

The torsal curves for the d_1'-congruence are obtained from (46) by changing the sign of θ_{10}.

Since M_1 does not contain $\sqrt{\theta_{10}}$, we have by subtracting this equation from (46) and dividing by a non-vanishing factor

$$16a'C'du^2 + \theta_4 dv^2 = 0, \tag{47}$$

a conjugate net on S. *It is the only conjugate net of the involution determined by the torsal curves of the d_1- and d_2-congruences.* There is a corresponding conjugate net relative to C_2 and C''. At any point on S the four tangents determined by these two nets are harmonically related if $2^8a'bCC' + \theta_4\theta_4' = 0$.

The focal points on d_1 are given by the factors of the covariant expression

$$L_1'\tau_2^2 - 2M_1'\tau_1\tau_2 + N_1'\tau_1^2, \tag{48}$$

where

$$L_1' = 2^8CD, \qquad 2M_1' = 2^8C(b_vD + E),$$

$$N_1' = 2^7b_vCE + C\theta_4'(D + 8a'T\sqrt{\theta_{10}}) + 2^6b_v^2CD - 2^5a'bC'\theta_9(\theta_{10} + 8a'C_u\sqrt{\theta_{10}}),$$

in which

$$T = 8a'b\theta_9 + 2C_uP + \theta_uC^2, \qquad D = 2^8a'^3bC_u\theta_9 + \theta_{10}P - 4a'T\sqrt{\theta_{10}},$$

$$E = a'bC\theta_9(\theta_4)_u - 2a'^2C\theta_4'\theta_{10} + a'C'\theta_{10}\sqrt{\theta_{10}}.$$

The focal points on d_1' are obtained from (48) by changing the sign of $\sqrt{\theta_{10}}$.

The guide curves for the d_1- and d_1'-congruences are given by $v =$ const. and these curves are not union curves.

The torsal curves, the focal points, and the guide curves relative to the complexes C_2 and C'' may be obtained at once by means of the transpositions (22) together with the transposition $(\theta_{10}, \theta_{10}')$, (θ_9, θ_9').

§ 7. The Congruences Determined by the Pair of Complexes C_1 and C_2.

The equations of the linear complexes C_1 and C_2, whose bilinear invariant (A, B) is zero, are given in (36) and (38). The following relations

$$a_{13}b_{24} = b_{12}a_{34}, \qquad a'^2\theta_{10}(b_{12}b_{34} - b_{14}^2) = b^2\theta_{10}'(a_{13}a_{24} - a_{14}^2), \tag{49}$$

between θ_{10} and θ_{10}' and the coefficients of (36) and (38) are useful in obtaining further results. As in the case of § 6 the equations of the two directrices of the congruence determined by C_1 and C_2 are given by

$$\begin{aligned} &- \epsilon a'b_{12}\sqrt{\theta_{10}}x_2 + ba_{13}\sqrt{\theta_{10}'}x_3 + (ba_{14}\sqrt{\theta_{10}'} - \epsilon a'b_{14}\sqrt{\theta_{10}})x_4 = 0, \\ &\epsilon a'b_{12}\sqrt{\theta_{10}}x_1 + (ba_{14}\sqrt{\theta_{10}'} + \epsilon a'b_{14}\sqrt{\theta_{10}})x_3 + (ba_{24}\sqrt{\theta_{10}'} - \epsilon a'b_{24}\sqrt{\theta_{10}})x_4 = 0, \end{aligned} \tag{50}$$

where $\epsilon = \pm 1$. Let δ_1 be the directrix determined by using $\epsilon = +1$ in

(50) and δ_2 the directrix corresponding to $\epsilon = -1$. It can be shown readily that *the points of intersection of δ_1 and δ_2 with the coördinate planes $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ are harmonic conjugates with respect to the osculating quadric Q.*

For the points τ_1 and τ_2 determining δ_1 we may take the points of intersection of δ_1 with the planes $x_2 = 0$ and $x_3 = 0$. The expressions for these points are

$$\begin{aligned}\tau_1 &= (ba_{24}\sqrt{\theta'_{10}} - a'b_{24}\sqrt{\theta_{10}})y \\ &\qquad - (ba_{14}\sqrt{\theta'_{10}} - a'b_{14}\sqrt{\theta_{10}})y_u - a'b_{12}\sqrt{\theta_{10}}y_{uv}, \\ \tau_2 &= -(ba_{34}\sqrt{\theta'_{10}} - a'b_{34}\sqrt{\theta_{10}})y \\ &\qquad + (ba_{14}\sqrt{\theta'_{10}} - a'b_{14}\sqrt{\theta_{10}})y_v - ba_{13}\sqrt{\theta'_{10}}y_{uv}.\end{aligned} \tag{51}$$

An examination of the elements entering into the equation of the torsal curves of the congruence determined by δ_1 shows that we obtain the same equation over again if we apply the transformation (22) to it. In this sense we may speak of the torsal curves of the δ_1-congruence as symmetric. Similar conditions hold for the δ_2-congruence. These nets may be represented by

$$\begin{aligned}&(\alpha_1\sqrt{\theta_{10}} + \alpha_2\sqrt{\theta'_{10}})du^2 + 2(\beta_1\sqrt{\theta_{10}} + \beta_2\sqrt{\theta'_{10}})dudv \\ &\qquad + (\gamma_1\sqrt{\theta_{10}} + \gamma_2\sqrt{\theta'_{10}})dv^2 = 0, \\ &(-\alpha_1\sqrt{\theta_{10}} + \alpha_2\sqrt{\theta'_{10}})du^2 + 2(-\beta_1\sqrt{\theta_{10}} + \beta_2\sqrt{\theta'_{10}})dudv \\ &\qquad + (-\gamma_1\sqrt{\theta_{10}} + \gamma_2\sqrt{\theta'_{10}})dv^2 = 0,\end{aligned} \tag{52}$$

where $\gamma_1, \gamma_2, \beta_2$ are obtained from $\alpha_2, \alpha_1, \beta_1$ respectively by (22).

If $\theta_{10} = 0$, $\theta'_{10} \neq 0$, the invariant of the complex C_1 is zero and hence C_1 is a special linear complex unless $\theta_{10} = 0$ by virtue of C being zero in which case the complex C_1 is indeterminate. Excluding that case we see that the congruence determined by C_1 and C_2 has two coincident straight line directrices.* This directrix is given by the two points whose coördinates are

$$(a_{24}, -a_{14}, 0, 0), \quad (a_{34}, 0, -a_{14}, a_{13}). \tag{53}$$

It is easy to verify that this line lies on the osculating quadric since $\theta_{10} = 0$. In the case we are considering we have but one congruence associated with the surface S instead of two. Its torsal curves may be obtained from (8) without a great deal of algebraic work. It is obvious geometrically that its guide curves are $v =$ const. since the directrix intersects the line yy_u.

If $\theta'_{10} = 0$, $C' \neq 0$, $\theta_{10} \neq 0$, we obtain a congruence generated by the line determined by the points whose coördinates are

$$(b_{34}, 0, -b_{14}, 0), \quad (b_{24}, -b_{14}, 0, b_{12}). \tag{54}$$

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* Second memoir, p. 163.

This line lies on the osculating quadric since $\theta'_{10} = 0$ and it intersects the line yy_v. Consequently the guide curves of the congruence determined by it are $u =$ const.

If $\theta_{10} = 0$, $\theta'_{10} = 0$, $C \neq 0$, $C' \neq 0$, the two complexes C_1 and C_2 are both special and the congruence determined by them degenerates into two systems of ∞^2 lines, viz., all the lines in the plane of the two axes of the special complexes and all the lines through their point of intersection.* This point of intersection is the point in which the lines (53) and (54) meet since in this case (53) and (54) are the two axes which are generators of different kinds on the osculating quadric.

Returning to the general case we find that the guide curves of the δ_1-congruence are given by

$$ba_{13}\sqrt{\theta'_{10}}du - a'b_{12}\sqrt{\theta_{10}}dv = 0, \tag{55}$$

and of the δ_2-congruence by

$$ba_{13}\sqrt{\theta'_{10}}du + a'b_{12}\sqrt{\theta_{10}}dv = 0. \tag{56}$$

Hence *the guide curves of the surface S with respect to these two congruences together constitute a projectively defined conjugate net on S, namely, the net*

$$a'^2C^2\theta'_{10}du^2 - b^2C'^2\theta_{10}dv^2 = 0. \tag{57}$$

* Loc. cit., p. 163.

LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH A CONTINUOUS INFINITUDE OF VARIABLES.*

By I. A. Barnett.

The purpose of this paper is to extend the theory of the linear partial differential equations

$$\frac{\partial F}{\partial \tau}(\tau; u_1, \cdots, u_n) + \sum_{i=1}^{n} f_i(\tau; u_1, \cdots, u_n)\frac{\partial F}{\partial u_i} = 0, \tag{1}$$

$$\sum_{i=1}^{n} f_i(u_1, \cdots, u_n)\frac{\partial F}{\partial u_i}(u_1, \cdots, u_n) = 0, \tag{2}$$

to equations which involve a continuous infinitude of variables. Since both of these equations involve the known functions $f_1, \cdots, f_n$ linearly, this suggests immediately the use of the Stieltjes integral for expressing the equations in the transcendental case. The equations to be studied are of the form

$$\frac{\partial F}{\partial \tau}[\tau, u] + \int_0^1 f[\xi, \tau, u] d_\xi \varphi[\xi, \tau, u] = 0 \tag{1'}$$

and

$$\int_0^1 f[\xi, u] d_\xi b[\xi, u] = 0, \tag{2'}$$

where ξ is a real variable, u is a continuous function of the variable ξ', f a given functional operation, F the functional sought, and φ and b stand for certain associated functionals of F. All of these symbols will be defined more precisely in § 1.

Equations similar to (1′) and (2′) but involving derivatives of functions of lines have already been studied by Volterra who considers the equation

$$\frac{\partial F[\tau, u]}{\partial \tau} + \int_0^1 f[\xi, \tau, u] F'[\xi, \tau, u] d\xi = 0, \tag{3}$$

where F' denotes the Volterra derivative of the functional F with respect to u.†

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* Presented to the Society in two papers, (1) at Chicago, December, 1918, and (2) at New York, April, 1920.

† Volterra, "Equazioni integro-differentiali ed equazioni alle derivate funzionali," *Atti della Reale Accademia dei Lincei* (1914), Vol. XXIII, serie 6, 1st semester, p. 551.

In § 1 some preliminary matters concerning differential equations involving functionals, implicit functional equations, and Fréchet differentials will be stated which will be found useful in the sequel. In § 2 solutions of (1′) will be shown to exist. This will come out as an application proved in a previous paper.* In § 3 there will be considered the question of finding all the solutions of a particular type. This will necessitate the use of an implicit functional equation studied by Lamson. Finally, in the last section analogous questions will be considered for the homogeneous equation (2′).

§ 1. Some Preliminary Lemmas.

The notations and definitions of the following lemma will be found in *Diff. Eq.* It is a condensation of Theorems I, III and IV.

LEMMA 1. *If the functional* $f[\xi, \tau, u]$ *is such that it possesses in the set*

$$(A_0) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \alpha, \qquad ||u - u_{00}|| \leqq \alpha$$

a difference function A, *satisfying*

$$f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u] = A[\xi, \eta, u, \bar{\eta}, \bar{u}; \bar{\eta} - \eta, \bar{u} - u],$$

which difference function besides having the linearity and modular properties designated by (2) *and* (3) [*Diff. Eq.*, § 2] *has also the modified continuity property* (1′); *then there exists a set of elements* B_0†

$$(B_0) \quad 0 \leqq \xi \leqq 1, \quad |\tau - \tau_0| \leqq \beta, \quad |\tau_0 - \tau_{00}| \leqq \beta, \quad ||u_0 - u_{00}|| \leqq \beta,$$

in which the unique solution $u = v[\xi, \tau, \tau_0, u_0]$ *of the functional equation*

$$(4) \qquad \frac{\partial u}{\partial \tau}(\xi, \tau) = f[\xi, \tau, u]$$

reducing to $u = u_0$ *for* $\tau = \tau_0$ *is defined and continuous with respect to all of its arguments and possesses a difference function* $B[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$.

One may simplify the statement of Lemma 2 by the following

Definition. The difference function $\Gamma[\xi, \tau, u, \bar{u}; \tilde{u}]$ is said to have a *reciprocal* for $\tau = \tau_0$, $u = \bar{u} = u_0$ if there exists a functional $\bar{\Gamma}[\xi, \tau, u, \bar{u}; \tilde{u}]$ with the properties (1), (2) and (3) of a difference function (*Diff. Eq.*, § 2) such that

$$\bar{\Gamma}[\xi, \tau_0, u_0, u_0; \Gamma[\xi, \tau_0, u_0, u_0; \tilde{u}]] = \tilde{u}(\xi)$$

* "Differential Equations Involving a Continuous Infinitude of Variables," this JOURNAL, Vol. XLIV, p. 172. This paper will be referred to as *Diff. Eq.*

† The set (B_0) is in a form somewhat different from that used in *Diff. Eq.* (§ 3) but it can always be taken in the form given here which for the purposes of the present paper is more convenient.

and with the further property that the vanishing of $\Gamma[\xi, \tau_0, u_0, u_0; \bar{u}]$ identically in ξ implies that $\bar{u}(\xi) = 0$ identically in ξ.

If now in Lamson's theorem* one interprets the range P to be the variable ξ, the class $\mathfrak{M}$ as the class of all continuous functions defined for the interval $0 \leqq \xi \leqq 1$, and the modulus $||u(\xi)||$ to be the maximum of the function u, the following lemma results:

LEMMA 2. *If, in the implicit functional equation*

$$G[\xi, \tau, u] = z(\xi), \tag{5}$$

the given functional G has the properties

(1) *Equation* (5) *is satisfied by the element* (τ_0, u_0, z_0),

(2) *G is real, single valued and continuous in its arguments,*

(3) *It has a difference function* $\Gamma[\xi, \tau, u, \bar{u}; \bar{u}]$ *for all* $(\xi, \tau, u, \bar{u})$ *in the set*

$$0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \sigma, \qquad ||u - u_0|| \leqq \sigma, \qquad ||\bar{u} - u_0|| \leqq \sigma,$$

(4) *For* $\tau = \tau_0$, $u = \bar{u} = u_0$, Γ *has a reciprocal* $\bar{\Gamma}$;

then there exists a constant $\sigma_1 \leqq \sigma$ such that the equation (5) *has one and only one solution*

$$u = H[\xi, \tau, z].$$

The functional H is uniformly continuous in its arguments and reduces to u_0 for $z = z_0$.

The lemma as stated is really not a special instance of Lamson's result since the left-hand side of equation (5) contains a parameter τ but with the hypotheses of Lemma 2 one could carry through step by step the proof given by Lamson and show that the conclusions of the lemma result.

Remark.—One could prove that the solution $H[\xi, \tau, z]$ has a difference function in its domain of definition. This proof could be effected by using Lemma 4, *Diff. Eq.* (§ 2), and a method of proof similar to Theorem IV of the same paper.

Suppose now that $u(\xi; \alpha)$ is for each fixed α of the interval $0 \leqq \alpha \leqq 1$ a continuous function of ξ and for each fixed ξ of $0 \leqq \xi \leqq 1$ a differentiable function of α. Suppose also that the functional $F[\xi, \tau, u]$ has a difference function $\Phi[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau}, \bar{u}]$ so that

$$F[\xi, \bar{\tau}, \bar{u}] - F[\xi, \tau, u] = \Phi[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau} - \tau, \bar{u} - u],$$

where $u = u(\xi'; \alpha)$, $\bar{u} = u(\xi'; \bar{\alpha})$ and Φ has the properties (1), (2) and (3). (*Diff. Eq.*, § 2.)

* K. Lamson, "A General Implicit Function Theorem," this JOURNAL, Vol. XLII, pp. 243–256.

Definition.—$\Phi[\xi, \tau, u, \tau, u; \bar{\tau}, \bar{u}]$ is called the differential of $F[\xi, \tau, u]$ at the element (τ, u) with the argument element $(\bar{\tau}, \bar{u})$.

LEMMA 3. *If a functional $F[\xi, \tau, u]$ has a difference function Φ, then $d/d\alpha\, F[\xi, \tau, u(\xi, \alpha)]$ is the differential at the element (ξ, τ, u) with the argument function $(0, \partial u/\partial\alpha)$.*

By hypothesis

$$\begin{aligned} F[\xi, \tau, u + \Delta u] - F[\xi, \tau, u] &\equiv F[\xi, \tau, u(\xi', \alpha + \Delta\alpha)] - F[\xi, \tau, u(\xi', \alpha)] \\ &= \Phi[\xi, \tau, u, \tau, u + \Delta u; 0, \Delta u] \end{aligned}$$

and hence by the linearity property of the difference function

$$\frac{F[\xi, \tau, u + \Delta u] - F[\xi, \tau, u]}{\Delta\alpha} = \Phi\left[\xi, \tau, u, \tau, u + \Delta u; 0, \frac{\Delta u}{\Delta\alpha}\right].$$

Thus by the continuity property of Φ it follows that as $\Delta\alpha \to 0$, the right-hand member tends to the expression $\Phi[\xi, \tau, u, \tau, u; 0, u_\alpha]$ as desired.

It would follow from Riesz's representation of a linear functional that the preceding lemma could also be written in the form

$$\frac{d}{d\alpha} F[\xi, \tau, u] = \int_0^1 \frac{\partial u(\xi', \alpha)}{\partial\alpha}\, d_{\xi'}\varphi[\xi, \xi', \tau, u]$$

and as Fréchet has pointed out there could be but one representation if it is specified that φ is identically zero for $\xi' = 1$ and that its discontinuities in ξ' are regular. This will always be supposed in what follows. The functional will be called the functional *associated* with the differential Φ of F or, more simply, the functional associated with F.

Corollary.—If the function $u(\xi, \alpha)$ is taken to be $u(\xi) + \alpha\bar{u}(\xi)$, then

$$\frac{d}{d\alpha} F[\xi, \tau, u + \alpha\bar{u}]\big|_{\alpha=0} = \int_0^1 \bar{u}(\xi')\, d_{\xi'}\varphi[\xi, \xi', \tau, u].$$

§ 2. Existence of Solutions of the Non-homogeneous Equation.

Consider the equation

$$\frac{\partial F}{\partial\tau}[\tau, u] + \int_0^1 f[\xi, \tau, u]\, d_\xi\varphi[\xi, \tau, u] = 0, \tag{1'}$$

where φ is the functional associated with F. One can now prove the following theorem.

THEOREM 1. *Suppose $v[\xi, \tau, \tau_0, u_0]$ is the unique solution of the functional equation*

$$\frac{\partial}{\partial\tau} u(\xi, \tau) = f[\xi, \tau, u],$$

of the kind described in Lemma 1, *then for each fixed* τ' *in* $|\tau' - \tau_{00}| \leqq \beta$, *the functional*

$$F[\xi, \tau, u] \equiv v[\xi, \tau', \tau, u] \tag{6}$$

is defined and continuous in the set of elements

$$0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \gamma, \qquad \|u - u_{00}\| \leqq \gamma,$$

where γ *is some positive number. Furthermore, F has a difference function* $\Phi[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau}, \bar{u}]$ *and satisfies equation* (1′).

Suppose

$$u = v[\xi, \tau, \tau', u']$$

is the solution of equation (4) passing through the initial element (τ', u'). Then it follows immediately from the uniqueness of the solution that

$$u' = v[\xi, \tau', \tau, u] = v[\xi, \tau', \tau, v[\xi, \tau, \tau', u']]. \tag{7}$$

Now, by Lemma 1, $v[\xi, \tau', \tau, u]$ possesses a difference function with respect to the arguments τ and u. It follows readily from this that $\partial v/\partial\tau$ exists. Hence, differentiating both sides of (7) with respect to τ, and making use of Lemma 3, one finds that

$$0 = \frac{\partial v}{\partial\tau}[\xi, \tau', \tau, u] + \bar{\Phi}\left[\xi, \tau, v, v; 0, \frac{\partial v}{\partial\tau}\right],$$

or

$$0 = \frac{\partial v}{\partial\tau}[\xi, \tau', \tau, u] + \int_0^1 \frac{\partial v}{\partial\tau}[\xi', \tau', \tau, u] d_{\xi'}\bar{\varphi}[\xi, \xi', \tau', \tau, u],$$

where $\bar{\varphi}$ is the functional associated with $v[\xi, \tau', \tau, u]$ when all but the last argument are kept fixed. But in view of (4) and (6) the last relation may be written

$$0 = \frac{\partial F}{\partial\tau}[\xi, \tau, u] + \int_0^1 f[\xi', \tau, u] d_{\xi'}\varphi[\xi, \xi', \tau, u],$$

where φ is the functional associated with F.

Corollary.—If the associated functional $\varphi[\xi, \xi', \tau, u]$ has a continuous derivative $\varphi_{\xi'}[\xi, \xi', \tau, u] \equiv \partial\varphi/\partial\xi'$, for all points of the interval $0 \leqq \xi' \leqq 1$, then there exists a solution of the equation

$$\frac{\partial F}{\partial\tau}[\xi, \tau, u] + \int_0^1 f[\xi', \tau, u] \frac{\partial\varphi}{\partial\xi'}[\xi, \xi', \tau, u] = 0.$$

This is in essence Volterra's result already cited, for the functional $\partial\varphi/\partial\xi'$

could be readily identified with the Volterra derivative of F with respect to u.

Example.—Consider the equation

$$\frac{\partial F}{\partial \tau}[\tau, u] + \int_0^1 \left\{ \int_0^1 K(\eta, \zeta)u(\zeta)d\zeta \right\} d_\eta \varphi[\eta, \tau, u] = 0, \tag{8}$$

where $K(\eta, \zeta)$ is symmetric. Here the functional $f[\xi, \tau, u]$ of equation (1′) is given by

$$f[\xi, \tau, u] = \int_0^1 K(\xi, \zeta)u(\zeta)d\zeta,$$

and the corresponding equation (4) is

$$\frac{\partial u}{\partial \tau}(\xi, \tau) = \int_0^1 K(\xi, \zeta)u(\zeta)d\zeta,$$

with the initial condition

$$u(\xi, \tau_0) = u_0(\xi).$$

Now it has been shown* that the unique solution of this system is given by

$$v[\xi, \tau] = u_0(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)(e^{(\tau-\tau_0)/\lambda_i} - 1) \int_0^1 \varphi_i(\zeta)u_0(\zeta)d\zeta,$$

where the λ_i are the characteristic numbers of K and the φ_i are the corresponding normed orthogonal characteristic functions. It follows therefore by the theory just developed that

$$F[\xi, \tau, u] \equiv u(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)(e^{-(\tau-\tau_0)/\lambda_i} - 1) \int_0^1 \varphi_i(\zeta)u(\zeta)d\zeta$$

is for every ξ of the interval (0, 1) a solution of equation (8). This could be readily verified directly.

§ 3. The General Solution of Equation (1′).

It is desired in this section to obtain a solution of equation (1′) in terms of which all others of a certain type are expressible. The following theorem is first proved.

Theorem 2. *If $F[\xi, \tau, u]$ is for every ξ of* (0, 1) *a solution of equation* (1′) *of the kind described in Theorem* 1, *then*

$$G[\tau, u] \equiv L[F[\xi, \tau, u]] \tag{9}$$

* See paper by writer, "Integro-differential Equations with the Constant Limits of Integration," *Bull. of Amer. Math. Soc.*, Vol. XXVI, pp. 193–203.

where L is an arbitrary functional eliminating the argument ξ and possessing for each fixed element (τ, u) of the set defined by the inequalities

$$|\tau - \tau_{00}| \leqq \beta, \qquad ||u - u_{00}|| \leqq \beta, \tag{10}$$

a difference function $\Lambda[F, F; F]$, is also a solution of equation (1′) *defined and continuous in* (10).

In the first place, one could readily verify that G has a difference function with respect to τ and u. This follows from the fact that both F and L possess difference functions with respect to their arguments. Hence it follows by the Corollary to Lemma 3 that

$$\frac{dG}{d\alpha}[\tau + \alpha, u + \alpha\bar{u}]|_{\alpha=0} = \frac{\partial G}{\partial \tau} + \int_0^1 \bar{u}(\xi) d_\xi \gamma[\xi, \tau, u],$$

where γ is the functional associated with G when the argument τ is kept fixed. But by (9) and Lemma 3 the left member of the preceding relation may be written

$$\frac{d}{d\alpha} L[F[\xi, \tau + \alpha, u + \alpha\bar{u}]]|_{\alpha=0} = \int_0^1 \frac{\partial F}{\partial \alpha}[\xi, \eta, \tau + \alpha, u + \alpha\bar{u}]|_{\alpha=0} d_\eta \lambda(\eta, F),$$

where λ is the functional associated with the differential $\Lambda[F, \bar{F}, \bar{\bar{F}}]$ of L. Applying again the Corollary of Lemma 3 to the last expression, one obtains finally

$$\begin{aligned} &\frac{\partial}{\partial \tau} G[\tau, u] + \int_0^1 \bar{u}(\xi) d_\xi \gamma[\xi, \tau, u] \\ &\qquad = \int_0^1 \left\{ \frac{\partial F}{\partial \tau}[\eta, \tau, u] + \int_0^1 \bar{u}(\xi) d_\xi \varphi[\xi, \eta, \tau, u] \right\} d_\eta \lambda(\eta, F). \end{aligned}$$

Substituting now $f[\xi, \tau, u]$ for $\bar{u}(\xi)$ and remembering that F is a solution of equation (1′), one sees that the right side of the last equation vanishes identically in τ and u, proving that

$$\frac{\partial G}{\partial \tau}[\tau, u] + \int_0^1 f[\xi, \tau, u] d_\xi \gamma[\xi, \tau, u] = 0$$

as desired.

On the basis of the implicit function theorem proved in Lemma 2 one is now able to give a method for expressing all solutions of a certain type. This is embodied in the following theorem:

Theorem 3. *If $F[\xi, \tau, u]$ is a solution of the kind described in Theorem* 1, *then any solution $L[\tau, u]$ of equation* (1′) *possessing a difference function $\Lambda[\tau, u, \bar{\tau}, \bar{u}; \bar{\bar{\tau}}, \bar{\bar{u}}]$ in the set of elements defined by* (10) *is a continuous functional of F when in F the element (τ, u) is thought of as fixed.*

Consider the functional equation

$$F[\xi, \tau, u] = z(\xi). \tag{11}$$

It is desired to show that this equation satisfies all the hypotheses of Lemma 2 and hence can be solved for u. In the first place it is clear from the definition (6) of F that equation (1′) is satisfied by $\tau = \tau_0$, $u = u_0$, $z = u_0$. Furthermore, since $F[\xi, \tau_0, u] \equiv u(\xi)$, it follows that the difference function of F has a reciprocal for $\tau = \tau_0$ and $u = \bar{u} = u_0$. Finally, F has all the required continuity properties. Hence, Lemma 2 is applicable and one may conclude that

$$u(\xi) = H[\xi, \tau, z], \tag{12}$$

where H is a continuous functional of its arguments for $0 \leqq \xi \leqq 1$, and for τ, z in a suitable neighborhood of $\tau = \tau_0$, $z = u_0$.

Substituting (12) in $L[\tau, u]$, one obtains

$$L[\tau, H[\xi, \tau, z]] \equiv M[\tau, z]$$

and it remains to show, in order to prove the theorem, that M does not involve τ explicitly. Proceeding as in the proof of the last theorem, one may obtain the relation

$$\frac{\partial L}{\partial \tau}[\tau, u] + \int_0^1 f[\eta, \tau, u] d_{\eta'}\lambda[\eta, \tau, u]$$
$$= \frac{\partial M}{\partial \tau} + \int_0^1 \left\{ \frac{\partial F}{\partial \tau} + \int_0^1 f[\xi, \tau, u] d_\xi \varphi[\xi, \eta, \tau, u] \right\} d_\eta \mu[\eta, F],$$

where μ is the functional associated with the differential of $M[\tau, u]$ which can be shown to exist since both L and H have difference functions (see Remark, Lemma 2). But L and F are by hypothesis solutions of equation (1′), so that

$$\frac{\partial M}{\partial \tau} = 0$$

as desired.

§ 4. The Homogeneous Equation.

Consider now the homogeneous equation

$$\int_0^1 f[\xi, u] d_\xi \sigma[\xi, u] = 0, \tag{2'}$$

where σ is the functional associated with the differential $\sum$ of the unknown functional $S[\xi, u]$.

Let $F[\xi, \tau, u]$ be a solution of equation (1′) of the kind described in Theorem 1. Consider an arbitrary functional $G[F[\xi, \tau, u]]$ of F. In other words, when the arguments τ, u are kept fixed in F, then for every F whose modulus lies between suitable limits, G yields a real number. This arbitrary functional will then depend upon τ and u and will be denoted by $H[\tau, u]$. Let it be assumed that $H[\tau, u]$ has the following properties:

(1) There exists an element (τ_0, u_0) for which

$$H[\tau_0, u_0] = 0.$$

(2) $H[\tau, u]$ as well as $H_\tau[\tau, u]$ are continuous functionals of their arguments for

$$|\tau - \tau_0| \leqq \epsilon, \qquad ||u - u_0|| \leqq \epsilon,$$

where ϵ is some positive number.

(3) $\left.\dfrac{\partial H}{\partial \tau}\right|_{\substack{\tau=\tau_0\\ u=u_0}} \neq 0.$

(4) $H[\tau, u]$ has a difference function in $|\tau - \tau_0| \leqq \epsilon$, $|\bar{\tau} - \tau_0| \leqq \epsilon$, $||u - u_0|| \leqq \epsilon$, $||\bar{u} - u_0|| \leqq \epsilon$.

It can be readily shown that such functionals H exist. For, it is clear from the defintion of $F[\xi, \tau, u]$ that

$$\left.\frac{\partial F}{\partial \tau}[\xi, \tau, u]\right|_{\substack{\tau=\tau_0\\ u=u_0}} = -\, u_0(\xi)$$

and excluding the case for which $u_0(\xi) \equiv 0$, it follows that if one takes

$$H[\tau, u] \equiv F[\xi_1, \tau, u] - u_0(\xi_1),$$

where ξ_1 is a value of ξ for which $u_0(\xi) \neq 0$, then the functional H will satisfy the conditions (1), (2), (3), (4).

Consider now the functional equation

$$H[\tau, u] = 0, \tag{13}$$

where $H \equiv G[F[\xi, \tau, u]]$ and has the properties (1), (2), (3), (4). If one applies to equation (13) a theorem proved by Bliss,* one may show that there exists a unique solution

$$\tau = K[u], \tag{14}$$

where K is a continuous functional of u in a neighborhood $||u - u_0|| \leqq \delta < \epsilon$ reducing for $u = u_0$ to $\tau = \tau_0$. It could be shown furthermore

* Bliss, *Transactions of the American Mathematical Society*, Vol. 21, April, 1920, p. 90.

that K has a difference function. The associated functional of the functional K will be denoted by $\kappa[u]$. Substituting (14) in $F[\xi, \tau, u]$, one obtains

$$F[\xi, K[u], u] \equiv S[\xi, u] \tag{15}$$

and it is desired to prove that S is a solution of the equation (2′).

In the first place it follows that since both K and F have difference functions with respect to their arguments, S has a difference function with respect to u. Let σ be the associated functional of the differential $\sum$ of S. By Corollary to Lemma 3 one obtains from (15) the relation

$$\int_0^1 \bar{u}(\xi)d_\xi\sigma[\xi, u] = \frac{\partial F}{\partial \tau}\int_0^1 \bar{u}(\xi)d_\xi\kappa[\xi, u] + \int_0^1 \bar{u}(\xi)d_\xi\varphi[\xi, K, u]. \tag{16}$$

But from

$$H[K[u], u] \equiv 0$$

one finds

$$\int \bar{u}(\xi)d_\xi\eta[\xi, K, u] + \frac{\partial H}{\partial \tau}\int_0^1 \bar{u}(\xi)d_\xi\kappa[\xi, u] = 0, \tag{17}$$

where η is the functional associated with the differential of H when the τ is kept fixed. Furthermore, since $H[\tau, u] \equiv G[F[\xi, \tau, u]]$, it follows by Theorem 2 that H is a solution of equation (1′); i.e.,

$$\frac{\partial H}{\partial \tau}[\tau, u] + \int_0^1 f[\xi, u]d_\xi\eta[\xi, K, u] = 0.$$

Hence, substituting for $\bar{u}(\xi)$ in (17) $f[\xi, u]$, and making use of the preceding relation, one finds that (17) reduces to

$$\int_0^1 f[\xi, u]d_\xi\kappa[\xi, u] = 1$$

at every element for which $\partial H/\partial\tau|_{\tau_0, u_0} \neq 0$. Substituting $f[\xi, u]$ for $\bar{u}(\xi)$ in (16) and making use of the last relation and of the fact that F is a solution of equation (1′), one obtains finally that

$$\int_0^1 f[\xi, u]d_\xi\sigma[\xi, u] = \frac{\partial F}{\partial \tau} - \frac{\partial F}{\partial \tau} = 0$$

as desired. The following theorem has therefore been proved:

Theorem 4. *If $F[\xi, \tau, u]$ is a solution of equation* (1′) *of the kind described in Theorem* 1 *and if G is an arbitrary functional of F*

$$G[F[\xi, \tau, u]] \equiv H[\tau, u]$$

such that H satisfies conditions (1), (2), (3) *and* (4), *then*

$$S[\xi, u] \equiv F[\xi, K[u], u]$$

is a solution of equation (2′) *where $K[u]$ is the unique solution of*

$$H[\tau, u] = 0$$

for τ.

Remark.—The arbitrary functional $H[\tau, u]$ should be chosen so that the implicit functional equation $H[\tau, u] = 0$ is easy to solve.

THEOREM 5. *If $L[u]$ is any solution of equation* (2′) *possessing a difference function $\Lambda[u, \bar{u}; \bar{u}]$, then L is a continuous functional of $S[\xi, u]$ described in Theorem* 4 *where u is thought of as fixed.*

In the first place it is readily shown that if $L[u]$ is a solution of equation (2′), then $L[F[\xi, \tau, u]]$ is independent of τ. For, by Lemma 3

$$\frac{dL}{d\tau} = \int_0^1 \frac{\partial F}{\partial \tau}[\xi, \tau, u] d_\xi \mu[\xi, u],$$

where μ is the functional associated with the differential of $L[F[\xi, \tau, u]] \equiv M[\tau, u]$ with respect to the second argument. But

$$\frac{\partial F}{\partial \tau}[\xi, \tau, u] = -f[\xi, \tau, u],$$

so that

$$\frac{dL}{d\tau} = -\int_0^1 f[\xi, \tau, u] d_\xi \mu[\xi, \tau, u] = 0,$$

since by hypothesis L is a solution of equation (2′).

Since $L[F[\xi, \tau, u]]$ is independent of τ, it follows that

$$L[F[\xi, \tau, u]] = L[u] \qquad \text{for} \qquad \tau = \tau_0,$$

and

$$L[F[\xi, \tau, u]] = L[S[\xi, u]] \qquad \text{for} \qquad \tau = K[u].$$

Hence it follows for all functions u of the type considered that

$$L[u] = L[S[\xi, u]].$$

This completes the proof of the theorem.

As an illustration of the results of this section, consider the equation associated with (8),

$$\text{(18)} \qquad \int_0^1 \left\{ \int_0^1 K[\eta, \zeta) u(\zeta) d\zeta \right\} d_\eta \varphi[\eta, u] = 0.$$

It has already been proved that a solution of equation (8) is given by

$$F[\xi, \tau, u] \equiv u(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)(e^{-(\tau-\tau_0)/\lambda_i} - 1) \int_0^1 \varphi_i(\zeta) u(\zeta) d\zeta.$$

Take for the arbitrary function $G[F]$, the expression

$$\int_0^1 \varphi_k(\xi)F[\xi,\,\tau,\,u]d\xi, \qquad \text{where} \qquad \int_0^1 \varphi_k(\xi)u(\xi)d\xi \neq 0\,*$$

and the function $H[\tau, u]$ is easily found to be

$$H[\tau,\,u] \equiv \int_0^1 \varphi_k(\xi)u(\xi)d\xi + (e^{-(\tau-\tau_0)/\lambda_k} - 1)\int_0^1 \varphi_k(\xi)u(\xi)d\xi$$
$$- \int_0^1 \varphi_k(\xi)u_0(\xi)d\xi = e^{-(\tau-\tau_0)/\lambda_k}\int_0^1 \varphi_k(\xi)u(\xi)d\xi - \int_0^1 \varphi_k(\xi)u_0(\xi)d\xi$$

so that

$$S[\xi,\,u] \equiv u(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)\int_0^1 \varphi_i(\zeta)u(\zeta)d\zeta \left[\left(\frac{\int_0^1 \varphi_k(\xi)u_0(\xi)}{\int_0^1 \varphi_k(\xi)u(\xi)}\right)^{\lambda_k/\lambda_i} - 1\right]^{\dagger}$$

is the "general solution" of equation (18).

As some particular solutions, one may take

$$\int_0^1 \varphi_j(\xi)S[\xi,\,u]d\xi = \left(\frac{\int_0^1 \varphi_k(\xi)u_0(\xi)d\xi}{\int_0^1 \varphi_k(\xi)u(\xi)d\xi}\right)^{\lambda_k/\lambda_j}\int_0^1 \varphi_j(\xi)u(\xi)d\xi, \qquad j \neq k.$$

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* It is well known in the theory of integral equations that this expression could not be zero for all the characteristic functions unless u were identically zero.

† Cf. p. 203 of article by writer entitled "Functionals Invariant under One-parameter Continuous Groups of Transformations in the Space of Continuous Functions," *Proceedings of the National Academy of Sciences*, Vol. 6, No. 4, pp. 200–204, April, 1920.

ON THE ORDERING OF THE TERMS OF POLARS AND TRANSVECTANTS OF BINARY FORMS.

By L. Isserlis.

The reduction of transvectants depends on the possibility of inserting between any two terms of a transvectant a series of others such that any two consecutive terms possess the property which is technically known as adjacence. It is asserted without proof, that this is possible, by Gordan ("Vorlesungen über Invariantentheorie," Zweiter Band, § 42, S. 44), by Clebsch (Binäre Formen, § 53, S. 185), and by Grace and Young ("Algebra of Invariants," § 50, p. 51).

In 1908 I gave a short sketch of a method of effecting the actual ordering of the terms, and in the present paper I develop one such method in full and illustrate it by ordering the 56 terms of the transvectant

$$T = \{(ax)^m(bx)^n(cx)^p,\ (dx)^s(ex)^t\}^3.$$

I gladly acknowledge my obligation to Professor M. J. M. Hill, F.R.S., of the University of London. He drew my attention to the importance of the problem in his lectures on the Algebra of Invariants and indeed himself devised a method of ordering the terms of a polar differing somewhat from that given in Section 2 below.

§ 1. Definitions and Notation.

Two terms of a polar are said to be adjacent when they differ only in that one has a factor of the form $(\alpha_h x)(\alpha_k y)$ while in the other this factor is replaced by $(\alpha_h y)(\alpha_k x)$.

Two terms of a transvectant are said to be adjacent when they differ merely in the arrangement of the letters in a pair of symbolical factors. Two terms can be adjacent in two ways:

(i) $P\cdot(\alpha_i\beta_j)(\alpha_h\beta_k)$ and $P\cdot(\alpha_i\beta_k)(\alpha_h\beta_j)$,

(ii) $P\cdot(\alpha_i\beta_j)(\alpha_h x)$ and $P\cdot(\alpha_h\beta_j)(\alpha_i x)$,

where the letters $\alpha_1, \alpha_2, \cdots$ belong to one of the two forms of the transvectant and the letters $\beta_1, \beta_2, \cdots$ belong to the other form, while P represents a symbolic product.

We have to discuss a method of arranging the terms of (i) the ordinary

polar

$$P = \{(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}\}_{y^r}$$
$$= \left(y\frac{\partial}{\partial x}\right)^r \{(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}\},$$

if we omit a numerical factor, where $y(\partial/\partial x) \equiv y_1(\partial/\partial x_1) + y_2(\partial/\partial x_2)$.

(ii) the mixed polar

$$Q = \left(y_1\frac{\partial}{\partial x}\right)^{r_1}\left(y_2\frac{\partial}{\partial x}\right)^{r_2}\cdots\left(y_m\frac{\partial}{\partial x}\right)^{r_m}\{(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}\}$$

and (iii) the transvectant

$$T = [(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p},\ (b_1x)^{m_1}(b_2x)^{m_2} \cdots (b_qx)^{m_q}]^r$$

in such a manner that each term shall be adjacent to its neighbours.

It is to be noted that the rth polar of

$$f = (a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}$$

is

$$\frac{(n_1 + n_2 + \cdots + n_p - r)!}{(n_1 + n_2 + \cdots + n_p)!}\cdot P.$$

and that

$$\frac{(n_1 + n_2 + \cdots + n_p - r_1 - r_2 - \cdots - r_m)!}{(n_1 + n_2 + \cdots + n_p)!}Q$$

is a mixed polar of f, but in what follows these numerical factors and the numerical coefficients of the various terms will be omitted, because their values do not affect the ordering of the terms.

§ 2. TO ORDER THE TERMS OF THE POLAR

$$P = \left(y\frac{\partial}{\partial x}\right)^r [(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}]$$

Let D_s denote an operator which polarizes powers of (a_sx) only, with regard to y, then

$$P = (D_1 + D_2 + \cdots + D_p)^r[(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}].$$

Consider the terms which arise from operators which are alike except that one factor D_s of the first is replaced in the other by D_t, thus

$$D_1^{r_1} D_2^{r_2} \cdots D_s^{\lambda} D_t^{\mu} \cdots D_p^{r_p} \tag{I}$$

and

$$D_1^{r_1} D_2^{r_2} \cdots D_s^{\lambda-1} D_t^{\mu+1} \cdots D_p^{r_p} \tag{II}$$

where $r_1 + r_2 + \cdots + \lambda + \mu + \cdots + r_p = r$. The terms of the polar arising from these are

$$T_1 = (a_1x)^{n_1-r_1}(a_1y)^{r_1} \cdots (a_sx)^{n_s-\lambda}(a_sy)^{\lambda}(a_tx)^{n_t-\mu}(a_ty)^{\mu} \cdots (a_px)^{n_p-r_p}(a_py)^{r_p},$$
$$T_2 = (a_1x)^{n_1-r_1}(a_1y)^{r_1} \cdots (a_sx)^{n_s-\lambda+1}(a_sy)^{\lambda-1}(a_tx)^{n_t-\mu-1}(a_ty)^{\mu+1} \cdots (a_px)^{n_p-r_p}(a_py)^{r_p},$$

and are adjacent since the factors $(a_sy)(a_tx)$ of the first are replaced by $(a_ty)(a_sx)$ in the second. Call the operators I and II consecutive operators; we must therefore arrange the terms of

$$(D_1 + D_2 + \cdots + D_p)^r$$

consecutively. Now it is obvious that the ordered development of $(D_1 + D_2)^r$ is $D_1^r,\ D_1^{r-1}D_2,\ D_1^{r-2}D_2^2,\ \cdots,\ D_2^r$. We shall denote this by $(D_1 + D_2)^r$ and then $(D_2 + D_1)^r$ will stand for $D_2^r,\ D_2^{r-1}D_1,\ \cdots,\ D_2D_1^{r-1},\ D_1^r$.

To obtain the ordered development of $(D_1 + D_2 + D_3)^r$ we first order $(D_1 + D_2)^r$. In the first term in which D_2 occurs, replace it by $D_2 + D_3$, in the second term in which it occurs replace it by $D_3 + D_2$, in the third by $D_2 + D_3$, and so on alternately. In this manner we obtain

$$D_1^r,\ D_1^{r-1}(D_2 + D_3),\ D_1^{r-2}(D_3 + D_2)^2,\ \cdots\ D_1^{r-2s}(D_3 + D_2)^{2s},\ D_1^{r-2s-1}(D_2 + D_3)^{2s+1},\ \cdots,$$

which becomes an ordered expansion of $(D_1 + D_2 + D_3)^r$ when each expression as $D_1^{r-2s}(D_3 + D_2)^{2s}$ has been expanded into a group of ordered terms of which the *last* is $D_1^{r-2s}D_2^{2s}$ and $D_1^{r-2s-1}(D_2 + D_3)^{2s+1}$ is expanded into an ordered group whose *first* term is $D_1^{r-2s-1}D_2^{2s+1}$ and this is consecutive to $D_1^{r-2s}D_2^{2s}$.

We shall prove by induction that this method is perfectly general. Let us assume that

$$(D_1 + D_2 + \cdots + D_p)^r$$

has been ordered. To order the terms of

$$(D_1 + D_2 + \cdots + D_p + D_{p+1})^r$$

replace D_p by $D_p + D_{p+1}$ in the first term of $(D_1 + D_2 + \cdots + D_p)^r$ in which it occurs, in the second term in which it occurs replace D_p by $D_{p+1} + D_p$, in the third by $D_p + D_{p+1}$ and so on alternately and we shall obtain the ordered expansion of

$$(D_1 + D_2 + \cdots + D_p + D_{p+1})^r.$$

For let

$$D_1^{\lambda_1}D_2^{\lambda_2} \cdots D_s^{\lambda_s}D_t^{\lambda_t} \cdots D_p^{\lambda_p} = T'_{-1},$$
$$D_1^{\lambda_1}D_2^{\lambda_2} \cdots D_s^{\lambda_s-1}D_t^{\lambda_t+1} \cdots D_p^{\lambda_p} = T_l$$

be two consecutive terms in the ordered expansion of $(D_1 + D_2 + \cdots + D_p)^r$ where $\sum\lambda = r$. When the substitutions of $D_p + D_{p+1}$ and $D_{p+1} + D_p$ for D_p have been made, then T_{l-1} and T_l become

$$T'_{l-1} = D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s} D_t^{\lambda_t} \cdots (D_p + D_{p+1})^{\lambda_p},$$
$$T'_l = D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s - 1} D_t^{\lambda_t + 1} \cdots (D_{p+1} + D_p)^{\lambda_p}.$$

On expanding the last factors we get two consecutively arranged groups of terms of the operator $(D_1 + D_2 + \cdots + D_p + D_{p+1})^r$. The last term of the first group is

$$D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s} D_t^{\lambda_t} \cdots D_{p+1}^{\lambda_p}.$$

The first term of the second group is

$$D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s - 1} D_t^{\lambda_t + 1} \cdots D_{p+1}^{\lambda_p}$$

and are themselves consecutive, so that the two groups together are a part of the ordered operator $(D_1 + D_2 + \cdots + D_{p+1})^r$. The above supposes that the suffixes s and t are neither of them equal to p, we must therefore consider specially the case in which T_{l-1} contains $D_s^{\lambda_s} D_p^{\lambda_p}$ and T_l contains $D_s^{\lambda_s - 1} D_p^{\lambda_p \pm 1}$.

Here

$$T_{l-1} = D_1^{\lambda_1} \cdots D_s^{\lambda_s} \cdots D_{p-1}^{\lambda_{p-1}} D_p^{\lambda_p},$$
$$T_l = D_1^{\lambda_1} \cdots D_s^{\lambda_s - 1} \cdots D_{p-1}^{\lambda_{p-1}} D_p^{\lambda_p \pm 1}.$$

After substitution

$$T'_{l-1} = D_1^{\lambda_1} \cdots D_s^{\lambda_s} \cdots D_{p-1}^{\lambda_{p-1}} (D_p + D_{p+1})^{\lambda_p},$$
$$T'_l = D_1^{\lambda_1} \cdots D_s^{\lambda_s - 1} \cdots D_{p-1}^{\lambda_{p-1}} (D_{p+1} + D_p)^{\lambda_p \pm 1}.$$

The last term of T'_{l-1} and the first term of T'_l are

$$D_1^{\lambda_1} \cdots D_s^{\lambda_s} \cdots D_{p-1}^{\lambda_{p-1}} D_{p+1}^{\lambda_p}$$

and

$$D_1^{\lambda_1} \cdots D_s^{\lambda_s - 1} \cdots D_{p-1}^{\lambda_{p-1}} D_{p+1}^{\lambda_p \pm 1},$$

and so these are consecutive.

Now $(D_1 + D_2)^r$ and $(D_1 + D_2 + D_3)^r$ have been ordered, hence $(D_1 + D_2 + D_3 + D_4)^r$ can be ordered, and so on. It is important in what follows to notice that when expanded in this way $(D_1 + D_2 + \cdots + D_p)^r$ commences with D_1^r and ends with D_k^r where k is one of the integers 2, 3, 4, $\cdots$ p and can be made any one of them we please.*

* The method adopted by Professor Hill for ordering $(D_1 + D_2 + \cdots + D_p)^r$ is as follows. The ordered arrangement of $(D_1 + D_2)^r$ is $D_1^r, D_1^{r-1}D_2, \cdots, D_2^r$, call this the direct order for $(D_1 + D_2)^r$ and $D_2^r, D_2^{r-1}D_1, \cdots, D_1^r$ the reverse order. A direct order for

§ 3. ON ORDERING THE TERMS OF A MIXED POLAR.

Let the mixed polar be

$$\left(y\frac{\partial}{\partial x}\right)^{r_k}_{y=y_k}\left(y\frac{\partial}{\partial x}\right)^{r_{k-1}}_{y=y_{k-1}}\cdots\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}\{(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}\}$$

and let the result of ordering

$$\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}\{(b_1x)^{m_1}\cdots(b_qx)^{m_q}\},$$

as in § 2, be $A_1 + A_2 + \cdots + A_{2s-1} + A_{2s} + \cdots$.

We may put

$$\left(y\frac{\partial}{\partial x}\right)^{r_2} = (D_1 + D_2 + \cdots + D_q)^{r_2}$$
$$= D_1^{r_2} + \cdots + D_t^{r_2}$$

when ordered as in § 2 where t is one of the numbers 2, 3, $\cdots$ q. Then the ordered development of

$$\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}\{(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}\}$$

is

$$(D_1^{r_2} + \cdots + D_t^{r_2})_{y=y_2}A_1 + (D_t^{r_2} + \cdots + D_1^{r_2})_{y=y_2}A_2 + \cdots$$
$$+ (D_1^{r_2} + \cdots + D_t^{r_2})_{y=y_2}A_{2s-1} + (D_t^{r_2} + \cdots + D_1^{r_2})_{y=y_2}A_{2s} + \cdots,$$

where $D_t^{r_2} + \cdots + D_1^{r_2}$ is $D_1^{r_2} + \cdots + D_t^{r_2}$ written in reverse order. To prove this we first observe that by § 2

$$(D_1^{r_2} + \cdots + D_t^{r_2})_{y=y_2}A_{2s-1}$$

is a group of consecutive terms; it will therefore be sufficient to prove that $D_t^{r_2}A_{2s-1}$ and $D_t^{r_2}A_{2s}$ are consecutive terms.

$(D_1 + D_2 + D_3)^r$ is

$(D_1 + D_2)^r$ in direct order $+$ [$(D_1 + D_2)^{r-1}$ in reverse order]D_3

$+$ [$(D_1 + D_2)^{r-2}$ in direct order]D_3^2 $+ \cdots$.

When the direct order for $(D_1 + D_2 + D_3)^r$ is established, we get a direct order for $(D_1 + D_2 + D_3 + D_4)^r$ by writing $(D_1 + D_2 + D_3)^r$ in direct order

$+$ [$(D_1 + D_2 + D_3)^{r-1}$ in reverse order]D_4
$+$ [$(D_1 + D_2 + D_3)^{r-2}$ in direct order]D_4^2

and so on.

It can be proved by induction that this method is general. From this can be deduced the ordering of a transvectant in which one of the two forms contains only a single binary f orm.

Now A_{2s-1} and A_{2s} are either of the form

$$A_{2s-1} = H(b_t x)^{m_t-\lambda_t}(b_t y_1)^{\lambda_t}(b_u x)(b_v y_1),$$
$$A_{2s} \;\; = H(b_t x)^{m_t-\lambda_t}(b_t y_1)^{\lambda_t}(b_v x)(b_u y_1),$$

or of the form indicated in the special case below, H being a symbolic product not involving $(b_t x)$.

Hence,

$$D_t^{r_2}A_{2s-1} = H(b_t x)^{m_t-\lambda_t-r_2}(b_t y_1)^{\lambda_t}(b_t y_2)^{r_2}(b_u x)(b_v y_1)$$

and

$$D_t^{r_2}A_{2s} = H(b_t x)^{m_t-\lambda_t-r_2}(b_t y_1)^{\lambda_t}(b_t y_2)^{r_2}(b_v x)(b_u y_1)$$

and are consecutive.

Special Case.—We must verify that this still holds in the special case in which t is u or v. We may, when $t = u$, write

$$A_{2s-1} = H(b_u x)^{m_u-\lambda_u}(b_u y_1)^{\lambda_u}(b_v y_1),$$
$$A_{2s} \;\; = H(b_u x)^{m_u-\lambda_u-1}(b_u y_1)^{\lambda_u+1}(b_v x),$$

so that

$$D_u^{r_2}A_{2s-1} = H(b_u x)^{m_u-\lambda_u-r_2}(b_u y_1)^{\lambda_u}(b_u y_2)^{r_2}(b_v y_1) = K(b_u x)(b_v y_1)$$

and

$$D_u^{r_2}A_{2s} = H(b_u x)^{m_u-\lambda_u-r_2-1}(b_u y_1)^{\lambda_u+1}(b_u y_2)^{r_2}(b_v x) = K(b_u y_1)(b_v x),$$

so that the adjacence holds in this case also.

Thus the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}(b_1 x)^{m_1}(b_2 x)^{m_2}\cdots(b_q x)^{m_q}$$

can be ordered; denote the result by

$$B_1 + B_2 + \cdots + B_{2s-1} + B_{2s} + \cdots.$$

Let $D_1^{r_3} + \cdots + D_k^{r_3}$ be the ordered development of the operator in $\left(y\frac{\partial}{\partial x}\right)^{r_3}_{y=y_3}$ or of $(D_1 + D_2 + \cdots + D_q)^{r_3}$ and let $D_k^{r_3} + \cdots + D_1^{r_3}$ be the same reversed.

Then the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_3}_{y=y_3}\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}(b_1 x)^{m_1}(b_2 x)^{m_2}\cdots(b_q x)^{m_q}$$

will be ordered if we expand it in the form

$$(D_1^{r_3} + \cdots + D_k^{r_3})_{y=y_3}B_1 + (D_k^{r_3} + \cdots + D_1^{r_3})_{y=y_3}B_2 + \cdots$$
$$+ (D_1^{r_3} + \cdots + D_k^{r_3})_{y=y_3}B_{2s-1} + (D_k^{r_3} + \cdots + D_1^{r_3})_{y=y_3}B_{2s} + \cdots.$$

As before, $(D_1^{r_3} + \cdots + D_k^{r_3})_{y=y_3}B_{2s-1}$ is a group of consecutive terms;

it will therefore be sufficient to prove that

$$D_k^{r_3} B_{2s-1} \quad \text{and} \quad D_k^{r_3} B_{2s}$$

are consecutive terms.

Now B_{2s-1} and B_{2s} are either of the form

$$\left.\begin{aligned} B_{2s-1} &= H(b_k x)^{m_k-\lambda_k-\mu_k}(b_k y_1)^{\lambda_k}(b_k y_2)^{\mu_k}(b_u x)(b_v y) \\ B_{2s} &= H(b_k x)^{m_k-\lambda_k-\mu_k}(b_k y_1)^{\lambda_k}(b_k y_2)^{\mu_k}(b_v x)(b_u y) \end{aligned}\right\} \text{ where } y \text{ is either } y_1 \text{ or } y_2$$

or of the form considered in the special case below. Therefore,

$$\begin{aligned} D_k^{r_3} B_{2s-1} &= H(b_k x)^{m_k-\lambda_k-\mu_k-r_3}(b_k y_1)^{\lambda_k}(b_k y_2)^{\mu_k}(b_k y_3)^{r_3}(b_u x)(b_v y), \\ D_k^{r_3} B_{2s} &= H(b_k x)^{m_k-\lambda_k-\mu_k-r_3}(b_k y_1)^{\lambda_k}(b_k y_2)^{\mu_k}(b_k y_3)^{r_3}(b_v x)(b_u y) \end{aligned}$$

and are consecutive.

Special Case.—The special case in which k is u or v can be treated as before; thus suppose k is u and that in $(b_u y)$, $(b_v y)$ y stands for y_2, then

$$\begin{aligned} B_{2s-1} &= H(b_u x)^{m_u-\lambda_u-\mu_u}(b_u y_1)^{\lambda_u}(b_u y_2)^{\mu_u}(b_v y_2), \\ B_{2s} &= H(b_u x)^{m_u-\lambda_u-\mu_u-1}(b_u y_1)^{\lambda_u}(b_u y_2)^{\mu_u+1}(b_v x), \end{aligned}$$

so that

$$D_k^{r_3} B_{2s-1} = H(b_u x)^{m_u-\lambda_u-\mu_u-r_3}(b_u y_1)^{\lambda_u}(b_u y_2)^{\mu_u}(b_u y_3)^{r_3}(b_v y_2) = K(b_u x)(b_v y_2)$$

and

$$D_k^{r_3} B_{2s} = H(b_u x)^{m_u-\lambda_u-\mu_u-r_3-1}(b_u y_1)^{\lambda_u}(b_u y_2)^{\mu_u+1}(b_u y_3)^{r_3}(b_v x) = K(b_u y_2)(b_v x),$$

so that the adjacence holds in this case.

We shall show that this method of ordering the mixed polar is perfectly general.

General Case.—Let the ordered development of the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_{k-1}}_{y=y_{k-1}} \cdots \left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2} \left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1} (b_1 x)^{m_1}(b_2 x)^{m_2} \cdots (b_q x)^{m_q}$$

be

$$P_1 + P_2 + \cdots + P_{2s-1} + P_{2s} + \cdots$$

and let

$$\left(y\frac{\partial}{\partial x}\right)^{r_k}_{y=y_k} \text{ or } (D_1 + D_2 + \cdots + D_q)^{r_k} \equiv D_1^{r_k} + \cdots + D_t^{r_k}$$

when ordered, $D_t^{r_k} + \cdots + D_1^{r_k}$ being the same written in reverse order. Then the ordered arrangement of the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_k}_{y=y_k} \left(y\frac{\partial}{\partial x}\right)^{r_{k-1}}_{y=y_{k-1}} \cdots \left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1} (b_1 x)^{m_1}(b_2 x)^{m_2} \cdots (b_q x)^{m_q}$$

is

$$(D_1^{r_k} + \cdots + D_t^{r_k})P_1 + (D_t^{r_k} + \cdots + D_1^{r_k})P_2 + \cdots$$
$$+ (D_1^{r_k} + \cdots + D_t^{r_k})P_{2s-1} + (D_t^{r_k} + \cdots + D_1^{r_k})P_{2s} + \cdots.$$

As before, we must prove that

$$D_t^{r_k}P_{2s-1} \quad \text{and} \quad D_t^{r_k}P_{2s}$$

are consecutive. Let

$$\begin{aligned} P_{2s-1} &= H(b_tx)^{m_t-\lambda_1-\lambda_2-\cdots-\lambda_{k-1}}(b_ty_1)^{\lambda_1}(b_ty_2)^{\lambda_2} \cdots (b_ty_{k-1})^{\lambda_{k-1}}(b_ux)(b_vy_i), \\ P_{2s} &= \text{------------------------------}(b_uy_i)(b_vx). \end{aligned}$$

Then

$$\begin{aligned} D_t^{r_k}P_{2s-1} &= H(b_tx)^{m_t-\lambda_1-\lambda_2-\cdots-\lambda_{k-1}-r_k}(b_ty_1)^{\lambda_1}(b_ty_2)^{\lambda_2} \cdots (b_ty_{k-1})^{\lambda_{k-1}}(b_ty_k)^{r_k} \\ &\qquad (b_ux)(b_vy_i), \\ D_t^{r_k}P_{2s} &= \text{------------------------------}(b_uy_i)(b_vx), \end{aligned}$$

and are consecutive.

In the special case in which u is t

$$\begin{aligned} P_{2s-1} &= H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}}(b_ty_1)^{\lambda_1}(b_ty_2)^{\lambda_2} \cdots (b_ty_i)^{\lambda_i} \cdots (b_ty_{k-1})^{\lambda_{k-1}}(b_vy_i), \\ P_{2s} &= H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}-1}(b_ty_1)^{\lambda_1} \cdots (b_ty_i)^{\lambda_i+1} \cdots (b_ty_{k-1})^{\lambda_{k-1}}(b_vx), \end{aligned}$$

so that

$$\begin{aligned} D_t^{r_k}P_{2s-1} &= H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}-r_k}(b_ty_1)^{\lambda_1} \cdots (b_ty_i)^{\lambda_i} \cdots (b_ty_{k-1})^{\lambda_{k-1}}(b_ty_k)^{r_k}(b_vy_i) \\ &= K(b_tx)(b_vy_i), \\ D_t^{r_k}P_{2s} &= H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}-r_k-1}(b_ty_1)^{\lambda_1} \cdots (b_ty_i)^{\lambda_i+1} \\ &\qquad \cdots (b_ty_{k-1})^{\lambda_{k-1}}(b_ty_k)^{r_k}(b_vx) = K(b_ty_i)(b_vx), \end{aligned}$$

so that the adjacence holds in this case. Hence the terms of a mixed polar can be ordered by the above method in the general case.

§ 4. THE ORDERING OF THE TERMS OF A TRANSVECTANT.

When it is required to calculate the transvectant

$$T = \{(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}, \quad (b_1x)^{m_1}(b_2x)^{m_2} \cdots (b_qx)^{m_q}\}^r$$

we first order the terms of the polar

$$\{(a_1x)^{n_1}(a_2x)^{n_2} \cdots (a_px)^{n_p}\}_{y^r}.$$

In the result replace y_1 by h_2 and y_2 by $-h_1$ where

$$(hx)^{m_1+m_2+\cdots+m_q} \equiv (b_1x)^{m_1}(b_2x)^{m_2} \cdots (b_qx)^{m_q}$$

and multiply the new result by $(hx)^{m_1+m_2+\cdots+m_p-r}$. This gives T when all the factors containing h have been evaluated. A factor such as $(a_1h)^{\lambda_1} \cdots (a_ph)^{\lambda_p}(hx)^{m_1+\cdots+m_q-r}$ is evaluated by polarizing $(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}$, λ_1 times with respect to y_1, λ_2 times with respect to $y_2, \cdots, \lambda_p$ times with respect to y_p and then putting

$$\left.\begin{matrix} y_{11} = -a_{12} \\ y_{12} = a_{11} \end{matrix}\right\} \left.\begin{matrix} y_{21} = -a_{22} \\ y_{22} = a_{21} \end{matrix}\right\} \cdots \begin{matrix} y_{p1} = -a_{p_2} \\ y_{p2} = a_{p_1} \end{matrix}$$

or symbolically

$$T = \sum (a_1x)^{n_1-\lambda_1} \cdots (a_px)^{n_p-\lambda_p}\left(y_1\frac{\partial}{\partial x}\right)^{\lambda_1} \cdots \left(y_p\frac{\partial}{\partial x}\right)^{\lambda_p}\{(b_1x)^{m_1}(b_2x)m^2\cdots(b_qx)^{m_q}\}$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_q = r.$$

Let D_{ks} be an operator which polarizes (b_sx) only, with respect to y_k, then we may put

$$\left(y_1\frac{\partial}{\partial x}\right)^{\lambda_1} = (D_{11} + D_{12} + \cdots + D_{1q})^{\lambda_1},$$

$$\left(y_2\frac{\partial}{\partial x}\right)^{\lambda_2} = (D_{21} + D_{22} + \cdots + D_{2q})^{\lambda_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(y_p\frac{\partial}{\partial x}\right)^{\lambda_p} = (D_{p1} + D_{p2} + \cdots + D_{pq})^{\lambda_p}.$$

T is thus arranged as a sum of groups of terms. The factors preceding the operators are already arranged so that if one is $H(a_sx)^u(a_tx)^v$ the next is $H(a_sx)^{u-1}(a_tx)^{v+1}$ reducible to $K(a_sx)$ and $K(a_tx)$ where H and K are products of linear factors only.

We must now develop the operators of form

$$\left(y\frac{\partial}{\partial x}\right)^{\lambda_1} \cdots y\frac{\partial}{\partial x}\Big)^{\lambda_p}$$

or

$$(D_{11} + D_{12} + \cdots + D_{1q})^{\lambda_1} \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_q}$$

so that when the operations have been carried out

(i) in each group every term is adjacent to its neighbors,

(ii) the last term of any group is adjacent to the first term of the next group.

§ 5. Let the transvectant

$$
\begin{aligned}
T = \cdots &+ H(a_s x)(D_{11} + \cdots + D_{1q})^{\lambda_1} \\
&\qquad\qquad \cdots (D_{s1} + \cdots + D_{sq})^{\lambda_s}(D_{t1} + \cdots + D_{tq})^{\lambda_t} \\
&\qquad\qquad \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} \\
&+ H(a_t x)(D_{11} + \cdots + D_{1q})^{\lambda_1} \\
&\qquad\qquad \cdots (D_{s1} + \cdots + D_{sq})^{\lambda_s+1}(D_{t1} + \cdots + D_{tq})^{\lambda_t-1} \\
&\qquad\qquad \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} \\
&+ \cdots,
\end{aligned}
$$

where H is a product of factors of type (ax). Denote the expanded forms of the operators in the terms written down above by $P_1 + P_2 + \cdots + P_\mu$ and $Q_1 + Q_2 + \cdots + Q_\nu$. Then

$$
\text{(i)} \qquad \left.\begin{array}{l} H(a_s x)\cdot P_\mu \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \\ H(a_t x)\cdot Q_1 \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \end{array}\right\} \text{must be adjacent terms of } T
$$

and

$$
\text{(ii)} \qquad \left.\begin{array}{l} H(a_s x)\cdot P_k \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \\ H(a_s x)\cdot P_{k+1} \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \end{array}\right\} \text{must be adjacent terms of } T.
$$

One way of satisfying (i) is to expand the operators so that the first operator $(D_{11} + \cdots + D_{1q})^{\lambda_1} \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p}$ always starts with $D_{11}^{\lambda_1}D_{21}^{\lambda_2} \cdots D_{p1}^{\lambda_p}$ (i.e., a product of D's with second suffix 1) and always ends with $D_{1q}^{\lambda_1} \cdots D_{sq}^{\lambda_s}D_{tq}^{\lambda_t} \cdots D_{pq}^{\lambda_p}$ (i.e., a product of D's with second suffix q). For then we may write the second operator as

$$
\begin{aligned}
(D_{1q} + \cdots + D_{11})^{\lambda_1} \cdots (D_{sq} + \cdots + D_{s1})^{\lambda_s+1}&(D_{tq} + \cdots + D_{t1})^{\lambda_t-1} \\
&\cdots (D_{pq} + \cdots + D_{p1})^{\lambda_p}
\end{aligned}
$$

so that the last term in the first group is

$$
H(a_s x)D_{1q}^{\lambda_1} \cdots D_{sq}^{\lambda_s}D_{tq}^{\lambda_t} \cdots D_{pq}^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\}_{y_1=a_1, y_2=a_2, \cdots}
$$

and the first operator in the second group is

$$
H(a_t x)D_{1q}^{\lambda_1} \cdots D_{sq}^{\lambda_s+1}D_{tq}^{\lambda_t-1} \cdots D_{pq}^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\}_{y_1=a_1, y_2=a_2, \cdots}
$$

and these are adjacent for they are of the form $K(a_sx)(b_qa_t)$ and $K(a_tx)(b_qa_s)$ where K is the same in both.

Condition (ii) can be satisfied in either of the following two ways. (a) If $P_k = \Delta Duv$ and $P_{k+1} = \Delta Duw$ where Δ is the same product of D's in both, or (b) if $P_k = \Delta DuvDtw$ and $P_{k+1} = \Delta DuwDtv$. For if (a) holds,

$$
H(a_sx)P_k\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} = K(b_wx)(b_va_u)
$$

and

$$
H(a_sx)P_{k+1}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} = K(b_vx)(b_wa_u),
$$

where K is the same in both. And if (b) holds,

$$H(a_s x)P_k\{(b_1 x)^{m_1} \cdots (b_q x)^{m_q}\} = K(b_v a_u)(b_w a_t)$$
$$H(a_s x)P_{k+1}\{(b_1 x)^{m_1} \cdots (b_q x)^{m_q}\} = K(b_w a_u)(b_v a_t),$$

where K is the same in both. Therefore in both cases we get adjacent terms of the transvectant.

§ 6. Thus we can order the terms of the transvectant if we can expand

$$(D_{11} + \cdots + D_{1q})^{\lambda_1}(D_{21} + \cdots + D_{2q})^{\lambda_2} \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p},$$

so that (i) it starts with $D_{11}^{\lambda_1}D_{21}^{\lambda_2} \cdots D_{p1}^{\lambda_p}$; (ii) it ends with $D_{1_q}^{\lambda_1}D_{2_q}^{\lambda_2} \cdots D_{p_q}^{\lambda_p}$; and (iii) any two consecutive terms are either of the form ΔDuv, ΔDuw or of the form $\Delta Duv\ Dtw, \Delta Duw Dtv$. We shall show later* that the general case can be deduced from the case $q = 2$. We shall therefore commence with this case. So we must arrange

$$(D_{11} + D_{12})^{\lambda_1}(D_{21} + D_{22})^{\lambda_2} \cdots (D_{p1} + D_{p2})^{\lambda_p},$$

so as to satisfy conditions (i), (ii), (iii) above. Omitting numerical coefficients we expand $(D_{11} + D_{12})^{\lambda_1}$ in the form $D_{11}^{\lambda_1} + D_{11}^{\lambda_1 - 1}D_{12} + D_{11}^{\lambda_1 - 2}D_{12}^2 + \cdots + D_{12}^{\lambda_1}$. We shall find it convenient to write for this

$$a_1 + a_2 + \cdots + a_{\lambda_1+1}.$$

Similarly, take

$$(D_{21} + D_{22})^{\lambda_2} = D_{21}^{\lambda_2} + D_{21}^{\lambda_2 - 1}D_{22} + \cdots + D_{22}^{\lambda_2}$$
$$= b_1 + b_2 + \cdots + b_{\lambda_2+1},$$

and so on. We are now concerned with the development of $(a_1 + a_2 + \cdots + a_{\lambda_1+1})(b_1 + b_2 + \cdots + b_{\lambda_2+1})(c_1 + c_2 + \cdots + c_{\lambda_3+1}) \cdots$ in such a manner that (i) it starts with $a_1b_1c_1 \cdots$; (ii) it ends with $a_{\lambda_1+1}b_{\lambda_2+1}c_{\lambda_3+1} \cdots$; and (iii) two consecutive terms are either of the form Pa_k and Pa_{k+1} where P is the same in both, or of the form Pa_kb_l and $Pa_{k+1}b_{l-1}$. Of course any of the letters $a, b, c, \cdots$ may be used instead of a or b. If the first form is used the adjacence of the corresponding terms of the transvectant is of the same kind as that of $(\alpha\beta)(\gamma x)\cdot K$ and $(\alpha\gamma)(\beta x)\cdot K$, but the second form produces terms whose adjacence is of the same kind as that of $(\alpha\beta)(\gamma\delta)K$ and $(\alpha\delta)(\gamma\beta)K$ for

$$a_k = D_{11}^{\lambda_1 - k+1}D_{12}^{k-1},$$
$$b_l = D_{12}^{\lambda_2 - l+1}D_{22}^{l-1},$$

so that if $Pa_k = MD_{11}$, then $Pa_{k+1} = MD_{12}$. But if $Pa_kb_l = MD_{11}D_{22}$, then $Pa_{k+1}b_{l-1} = MD_{12}D_{21}$, showing that condition (iii) is satisfied.

* § 9.

First let all the exponents or all the exponents but one be even (in the latter case put the factor $(D_{s1} + D_{s2})^{\lambda_s}$ say, with the odd exponent λ_s, first). Multiply out by the rule

$$\begin{aligned}(u + v + \cdots + x + y)(p + q + r + \cdots)\\ &= (u + v + \cdots + x + y)p + (y + x + \cdots + v + u)q\\ &\qquad + (u + v + \cdots + x + y)r + \cdots\\ &= up + vp + \cdots + yp + yq + xq + \cdots + uq + ur + vr + \cdots,\end{aligned}$$

and if there are more than two factors, the first two are to be multiplied together by the rule and then the result is to be multiplied by the third factor in accordance with the rule and so on. Thus

$$\begin{aligned}&(a_1 + a_2 + \cdots + a_{\lambda_1+1})(b_1 + b_2 + \cdots + b_{\lambda_2+1})(c_1 + c_2 + \cdots + c_{\lambda_3+1})\\ &\quad = [(a_1 + a_2 + \cdots + a_{\lambda_1+1})b_1 + (a_{\lambda_1+1} + \cdots + a_1)b_2\\ &\qquad + \cdots + (a_1 + \cdots + a_{\lambda_1+1})b_{\lambda_2+1}](c_1 + \cdots + c_{\lambda_3+1})\end{aligned}$$

in which b_{λ_2+1} multiplies the a's in direct order since $\lambda_2 + 1$ is odd. The complete product is

$$\begin{aligned}&(a_1b_1 + \cdots + a_{\lambda_1+1}b_{\lambda_2+1})c_1 + (a_{\lambda_1+1}b_{\lambda_2+1} + \cdots + a_1b_1)c_2 + \cdots\\ &\qquad + (a_1b_1 + \cdots + a_{\lambda_1+1}b_{\lambda_2+1})c_{\lambda_3+1},\end{aligned}$$

ending thus since $\lambda_3 + 1$ is odd.

It is clear that with any number of factors this rule will give a product commencing with $a_1b_1c_1 \cdots$ and ending with $a_{\lambda_1+1}b_{\lambda_2+1}c_{\lambda_3+1} \cdots$ and any two consecutive terms will be of the form Pa_k, Pa_{k+1}.

§ 7. The case in which several of the exponents are odd cannot be done so simply. It is sufficient to consider the expansion of

$$(a_1 + \cdots + a_{\lambda_1+1})(b_1 + \cdots + b_{\lambda_2+1})(c_1 + c_2 + \cdots + c_{\lambda_3+1}) \cdots,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are *all* odd, for when this has been multiplied out so as to satisfy the conditions (i), (ii), (iii) of § 6, the product can be multiplied in turn by each of the factors containing an odd number of terms without any difficulty.

So we consider

$$(a_1 + \cdots + a_{2l})(b_1 + \cdots + b_{2m})(c_1 + c_2 + \cdots + c_{2n}) \cdots,$$

where there is an even number of terms in each factor.

Now $(a_1 + a_2)(b_1 + b_2)$ may be expanded by the scheme

$$a_1\begin{cases} b_1\\ b_2\end{cases}$$

$$a_2\begin{cases} b_1\\ b_2\end{cases}$$

i.e., $a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$.

This satisfies the conditions (i), (ii), (iii) and the two middle terms will ultimately produce terms of the transvectant whose adjacence is of the same kind as that of $K(\alpha\beta)(\gamma\delta)$ and $K(\alpha\delta)(\gamma\beta)$. For $(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)$ we use the scheme

$$a_1 \left\{ \begin{array}{l} b_1 \left\{ \begin{array}{l} c_1 \\ c_2 \end{array} \right. \\ b_2 \left\{ \begin{array}{l} c_2 \\ c_1 \end{array} \right. \end{array} \right.$$

$$a_2 \left\{ \begin{array}{l} b_1 \left\{ \begin{array}{l} c_1 \\ c_2 \end{array} \right. \\ b_2 \left\{ \begin{array}{l} c_1 \\ c_2 \end{array} \right. \end{array} \right.$$

or $a_1b_1c_1 + a_1b_1c_2 + a_1b_2c_2 + a_1b_2c_1 + a_2b_1c_1 + a_2b_1c_2 + a_2b_2c_1 + a_2b_2c_2$ where again all the conditions are satisfied and the adjacence of the middle terms is as in the previous case.

Similarly

$$(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)(d_1 + d_2)$$

is arranged by the scheme

$$a_1 \left\{ \begin{array}{l} b_1 \left\{ \begin{array}{l} c_1 \left\{ \begin{array}{l} d_1 \\ d_2 \end{array} \right. \\ c_2 \left\{ \begin{array}{l} d_2 \\ d_1 \end{array} \right. \end{array} \right. \\ b_2 \left\{ \begin{array}{l} c_2 \left\{ \begin{array}{l} d_1 \\ d_2 \end{array} \right. \\ c_1 \left\{ \begin{array}{l} d_2 \\ d_1 \end{array} \right. \end{array} \right. \end{array} \right.$$

$$a_2 \left\{ \begin{array}{l} b_1 \left\{ \begin{array}{l} c_1 \left\{ \begin{array}{l} d_1 \\ d_2 \end{array} \right. \\ c_2 \left\{ \begin{array}{l} d_2 \\ d_1 \end{array} \right. \end{array} \right. \\ b_2 \left\{ \begin{array}{l} c_1 \left\{ \begin{array}{l} d_1 \\ d_2 \end{array} \right. \\ c_2 \left\{ \begin{array}{l} d_1 \\ d_2 \end{array} \right. \end{array} \right. \end{array} \right.$$

and so on, where in any vertical column (say the fourth) the entries d_1, d_2 and d_2, d_1 occur alternately *except that the last two entries are alike.*

§ 8. Now consider the product

$$(a_1 + a_2 + a_3 + a_4)(b_1 + b_2 + b_3 + b_4) = AB.$$

Bracket the last two terms in each bracket together, thus:

$$AB = (a_1 + a_2 + \overline{a_3 + a_4})(b_1 + b_2 + \overline{b_3 + b_4}),$$

and so begin by treating the product as though the factors contained an odd number of terms. So

$$\begin{aligned} AB &= b_1(a_1 + a_2 + \overline{a_3 + a_4}) + b_2(\overline{a_4 + a_3} + a_2 + a_1) \\ &\qquad\qquad + (b_3 + b_4)(a_1 + a_2 + \overline{a_3 + a_4}) \\ &= b_1a_1 + b_1a_2 + b_1a_3 + b_1a_4 + b_2a_4 + b_2a_3 + b_2a_2 + b_2a_1 \\ &\qquad\qquad + a_1(b_3 + b_4) + a_2(b_4 + b_3) + (a_3 + a_4)(b_3 + b_4). \end{aligned}$$

Now we can deal with $(a_3 + a_4)(b_3 + b_4)$. It will commence with a_3b_3 which will be " consecutive " to a_2b_3, where we use the word consecutive for brevity to denote that these operator terms are correctly ordered for producing adjacent terms of the transvectant, and it will end with a_4b_4.

In the same way

$$\begin{aligned} &(a_1 + a_2 + \overline{a_3 + a_4})(b_1 + b_2 + \overline{b_3 + b_4})(c_1 + c_2 + \overline{c_3 + c_4}) \\ &\qquad = [b_1a_1 + 10 \text{ ordered terms} + a_2b_3 + a_3a_4(b_3 + b_4)]c_1 \\ &\qquad\quad + [(b_4 + b_3)(a_4 + a_3) + a_2b_3 + 10 \text{ ordered terms} + b_1a_1]c_2 \\ &\qquad\quad + a_1b_1(c_3 + c_4) + b_1a_2(c_4 + c_3) + \cdots + a_2b_3(c_4 + c_3) \\ &\qquad\quad + (c_3 + c_4)(b_3 + b_4)(a_3 + a_4) \end{aligned}$$

can be consecutively arranged, and starts with $a_1b_1c_1$ and ends with $a_4b_4c_4$.

This can be generalized.

$$\begin{aligned} &(a_1 + a_2 + \cdots + a_{2p-2} + \overline{a_{2p-1} + a_{2p}})(b_1 + b_2 + \cdots + b_{2q-2} + \overline{b_{2q-1} + b_{2q}}) \\ &\quad = b_1(a_1 + \cdots + a_{2p}) + b_2(a_{2p} + \cdots + a_1) + \cdots + b_{2q-2}(a_{2p} + \cdots + a_1) \\ &\qquad + (b_{2q-1} + b_{2q})(a_1 + a_2 + \cdots + a_{2p-2} + \overline{a_{2p-1} + a_{2p}}). \end{aligned}$$

The last product is

$$\begin{aligned} &a_1(b_{2q-1} + b_{2q}) + a_2(b_{2q} + b_{2q-1}) + \cdots + a_{2p-2}(b_{2q} + b_{2q-1}) \\ &\qquad\qquad + (b_{2q-1} + b_{2q})(a_{2p-1} + a_{2p}) \end{aligned}$$

and therefore ends with $a_{2p}b_{2q}$.

Similarly

$$\begin{aligned} &(a_1 + a_2 + \cdots + a_{2p})(b_1 + \cdots + b_{2q})(c_1 + c_2 + \cdots + c_{2r-2} + c_{2r-1} + c_{2r}) \\ &\quad = (a_1b_1 + \cdots + a_{2p-2}b_{2q-1} + \overline{b_{2q-1} + b_{2q}}\;\overline{a_{2p-1} + a_{2p}})(c_1 + \cdots + c_{2r}) \\ &\quad = c_1(a_1b_1 + \cdots + a_{2p-2}b_{2q-1} + \cdots + b_{2q}a_{2p}) \\ &\qquad + c_2 \text{ (same reversed)} \\ &\qquad + c_3 \text{ (same direct)} \\ &\qquad + \cdots \\ &\qquad + c_{2r-2} \text{ (same reversed)} \\ &\qquad + (c_{2r-1} + c_{2r})(a_1b_1 + \cdots + a_{2p-2}b_{2q-1} + \overline{b_{2q-1} + b_{2q}}\;\overline{a_{2p-1} + a_{2p}}) \end{aligned}$$

and the last line is

$$a_1b_1(c_{2r-1}+c_{2r})+\cdots+a_{2p-2}b_{2q-1}(c_{2r}+c_{2r-1})$$
$$+(c_{2r-1}+c_{2r})(b_{2q-1}+b_{2q})(a_{2p-1}+a_{2p});$$

therefore when the product in the preceding line is ordered, the whole product now under consideration will be ordered and starts with $a_1b_1c_1$ and ends with $a_{2p}b_{2q}c_{2r}$.

§ 9. We can therefore in all cases " order " the operator

$$A=(D_{11}+D_{12})^{\lambda_1}(D_{21}+D_{22})^{\lambda_2}\cdots(D_{p1}+D_{p2})^{\lambda_p}.$$

The transition from this to the operator

$$B=(D_{11}+D_{12}+D_{13})^{\lambda_1}(D_{21}+D_{22}+D_{23})^{\lambda_2}\cdots(D_{p1}+D_{p2}+D_{p3})^{\lambda_p}$$

is effected as follows:

In the *last* term of A each term of the form D_{k2} is replaced by $D_{k2}+D_{k3}$,
In the term before by $D_{k3}+D_{k2}$,
In the term before that by $D_{k2}+D_{k3}$,

and so on where $k=1, 2, \cdots p$.

Then since A begins with $D_{11}^{\lambda_1}\cdots D_{p1}^{\lambda_p}$ and ends with $D_{12}^{\lambda_1}\cdots D_{p2}^{\lambda_p}$, therefore B begins with $D_{11}^{\lambda_1}\cdots D_{p1}^{\lambda_p}$ and ends with $(D_{12}+D_{13})^{\lambda_1}\cdots(D_{p2}+D_{p3})^{\lambda_p}$, i.e., B ends with $D_{13}^{\lambda_1}\cdots D_{p3}^{\lambda_p}$.

Let two consecutive terms in A be

$$\Delta D_{12}^{\mu_1}D_{22}^{\mu_2}\cdots D_{p2}^{\mu_p} \quad\text{and}\quad \Delta' D_{12}^{\mu_1'}D_{22}^{\mu_2'}\cdots D_{p2}^{\mu_p'}$$ where Δ, Δ'

are products of D's whose second suffix is 1. These become in B say,

$$\Delta(D_{12}+D_{13})^{\mu_1}(D_{22}+D_{23})^{\mu_2}\cdots(D_{p2}+D_{p3})^{\mu_p}$$

and

$$\Delta'(D_{13}+D_{12})^{\mu_1'}(D_{22}+D_{23})^{\mu_2'}\cdots(D_{p3}+D_{p2})^{\mu_p'},$$

of which the first ends with $\Delta D_{13}^{\mu_1}\cdots D_{p3}^{\mu_p}$ and the second begins with $\Delta' D_{13}^{\mu_1'}\cdots D_{p3}^{\mu_p'}$ and these are " consecutive " since the original terms were so.

Similarly when $(D_{11}+D_{12}+D_{13})^{\lambda_1}\cdots(D_{p1}+D_{p2}+D_{p3})^{\lambda_p}$ has been expanded so as to satisfy the conditions (i), (ii), (iii) we can deduce the correct expansion of

$$(D_{11}+D_{12}+D_{13}+D_{14})^{\lambda_1}\cdots(D_{p1}+D_{p2}+D_{p3}+D_{p4})^{\lambda_p}$$

as follows:

In the last term in which D_{k3} occurs replace it by $D_{k3} + D_{k4}$,
In the term before replace it by $D_{k4} + D_{k3}$,
In the term before that replace it by $D_{k3} + D_{k4}$,

and so on, and we obtain an ordered expansion starting with $D_{11}^{\lambda_1}D_{21}^{\lambda_2} \cdots D_{p1}^{\lambda_p}$ and ending with $D_{14}^{\lambda_1}D_{24}^{\lambda_2} \cdots D_{p4}^{\lambda_p}$.

It is evident that the method is perfectly general, and we thus succeed in ordering the terms of the transvectant of any two binary forms.

As an illustration we will order the 56 terms of

$$T = \{(ax)^m(bx)^n(cx)^p,\ (dx)^s(ex)^t\}^3.$$

§ 10. Let

$$T = \{(ax)^m(bx)^n(cx)^p,\ (dx)^s(ex)^t\}^3.$$

The polarizing operator is $(D_1 + D_2 + D_3)^3$. Now

$$(D_3 + D_2)^3 = D_3^3 + D_3^2D_2 + D_2^2D_3 + D_2^3;$$

$$\therefore\ (D_3 + D_2 + D_1)^3 = D_3^3 + D_3^2(D_2 + D_1) + D_3(D_1 + D_2)^2 + (D_2 + D_1)^3,$$

reversing this, we find

$$\begin{aligned}(D_1 + D_2 + D_3)^3 = D_1^3 + D_1^2D_2 + D_1D_2^2 + D_2^3 + D_2^2D_3 + D_2D_1D_3 \\ + D_1^2D_3 + D_1D_3^2 + D_2D_3^2 + D_3^3.\end{aligned}$$

Let D_{11} polarize (dx) with regard to y_1	Let D_{12} polarize (ex) with regard to y_1
D_{21} ——— y_2	D_{22} ——— y_2
D_{31} ——— y_3	D_{23} ——— y_3

Also let

$$\begin{array}{rl|rl|rl} y_{11} = & a_2 & y_{21} = & b_2 & y_{31} = & c_2 \\ y_{12} = & -a_1 & y_{22} = & -b_1 & y_{32} = & -c_1 \end{array}$$

Then

$$\begin{aligned}
T = {} & (ax)^{m-3}(bx)^n(cx)^p(D_{11} + D_{12})^3\{(dx)^s(ex)^t\} && 1\\
& + (ax)^{m-2}(bx)^{n-1}(cx)^p(D_{22} + D_{21})(D_{12} + D_{11})^2\{(dx)^s(ex)^t\} && 2\\
& + (ax)^{m-1}(bx)^{n-2}(cx)^p(D_{11} + D_{12})(D_{21} + D_{22})^2\{(dx)^s(ex)^t\} && 3\\
& + (ax)^m(bx)^{n-3}(cx)^p(D_{22} + D_{21})^3\{(dx)^s(ex)^t\} && 4\\
& + (ax)^m(bx)^{n-2}(cx)^{p-1}(D_{31} + D_{32})(D_{21} + D_{22})^2\{(dx)^s(ex)^t\} && 5\\
& + (ax)^{m-1}(bx)^{n-1}(cx)^{p-1}(D_{12}+D_{11})(D_{22}+D_{21})(D_{32}+D_{31})\{(dx)^s(ex)^t\} && 6\\
& + (ax)^{m-2}(bx)^n(cx)^{p-1}(D_{31} + D_{32})(D_{11} + D_{12})^2\{(dx)^s(ex)^t\} && 7\\
& + (ax)^{m-1}(bx)^n(cx)^{p-2}(D_{12} + D_{11})(D_{32} + D_{31})^2\{(dx)^s(ex)^t\} && 8\\
& + (ax)^m(bx)^{n-1}(cx)^{p-2}(D_{21} + D_{22})(D_{31} + D_{32})^2\{(dx)^s(ex)^t\} && 9\\
& + (ax)^m(bx)^n(cx)^{p-3}(D_{32} + D_{31})^3\{(dx)^s(ex)^t\}. && 10
\end{aligned}$$

Thus

$$T = T_1 + T_2 + \cdots + T_9 + T_{10}.$$

$$\begin{aligned}
&\text{The operator in } T_1 \text{ is } D_{11}^3 + D_{11}^2 D_{12} + D_{11} D_{12}^2 + D_{12}^3 \\
&\text{in } T_2 \text{ is } D_{12}^2(D_{22} + D_{21}) + D_{11}D_{12}(D_{21} + D_{22}) + D_{11}^2(D_{11} + D_{12}) \\
&\text{in } T_3 \text{ is } D_{21}^2(D_{11} + D_{12}) + D_{21}D_{22}(D_{12} + D_{11}) + D_{22}^2(D_{11} + D_{12}) \\
&\text{in } T_4 \text{ is } D_{22}^3 + D_{22}^2 D_{21} + D_{22} D_{21}^2 + D_{21}^3 \\
&\text{in } T_5 \text{ is } D_{21}^2(D_{31} + D_{32}) + D_{21}D_{22}(D_{32} + D_{31}) + D_{22}^2(D_{31} + D_{32}) \\
&\text{in } T_6 \text{ is } D_{12}D_{22}D_{32} + D_{12}D_{22}D_{31} + D_{12}D_{21}D_{32} + D_{12}D_{21}D_{31} \\
&\qquad + D_{11}D_{22}D_{31} + D_{11}D_{22}D_{32} + D_{11}D_{21}D_{32} + D_{11}D_{21}D_{31} \\
&\text{in } T_7 \text{ is } D_{11}^2(D_{31} + D_{32}) + D_{11}D_{12}(D_{32} + D_{31}) + D_{12}^2(D_{31} + D_{32}) \\
&\text{in } T_8 \text{ is } D_{32}^2(D_{12} + D_{11}) + D_{32}D_{31}(D_{11} + D_{12}) + (D_{12} + D_{11})D_{31}^2 \\
&\text{in } T_9 \text{ is } D_{31}^2(D_{21} + D_{22}) + D_{31}D_{32}(D_{22} + D_{21}) + D_{32}^2(D_{21} + D_{22}) \\
&\text{in } T_{10} \text{ is } D_{32}^3 + D_{32}^2 D_{31} + D_{32} D_{31}^2 + D_{31}^3.
\end{aligned}$$

Performing the indicated polarizations and remembering that $(\alpha y_1)_{y_{11}=a_2,\, y_{12}=-a_1}$ is (αa) we easily obtain the following development:

$$\begin{aligned}
T_1 = (ax)^{m-3}(bx)^n(cx)^p\{&(dx)^{s-3}(ex)^t(da)^3 + (dx)^{s-2}(ex)^{t-1}(da)^2(ea) \\
&+ (dx)^{s-1}(ex)^{t-2}(da)(ea)^2 + (dx)^s(ex)^{t-3}(ea)^3\},
\end{aligned}$$

$$\begin{aligned}
T_2 = (ax)^{m-2}(bx)^{n-1}(cx)^p\{&(dx)^s(ex)^{t-3}(eb)(ea)^2 + (dx)^{s-1}(ex)^{t-2}(db)(ea)^2 \\
&+ (dx)^{s-2}(ex)^{t-1}(da)(db)(ea) + (dx)^{s-1}(ex)^{t-2}(da)(ea)(eb) \\
&+ (dx)^{s-2}(ex)^{t-1}(eb)(da)^2 + (dx)^{s-3}(ex)^t(db)(da)^2\},
\end{aligned}$$

$$\begin{aligned}
T_3 = (ax)^{m-1}(bx)^{n-2}(cx)^p\{&(dx)^{s-3}(ex)^t(da)(db)^2 + (dx)^{s-2}(ex)^{t-1}(ea)(db)^2 \\
&+ (dx)^{s-1}(ex)^{t-2}(db)(ea)(eb) + (dx)^{s-2}(ex)^{t-1}(da)(db)(eb) \\
&+ (dx)^{s-1}(ex)^{t-2}(da)(eb)^2 + (dx)^s(ex)^{t-3}(ea)(eb)^2\},
\end{aligned}$$

$$\begin{aligned}
T_4 = (ax)^m(bx)^{n-3}(cx)^p\{&(dx)^s(ex)^{t-3}(eb)^3 + (dx)^{s-1}(ex)^{t-2}(eb)^2(db) \\
&+ (dx)^{s-2}(ex)^{t-1}(eb)(db)^2 + (ex)^t(dx)^{s-3}(db)^3\},
\end{aligned}$$

$$\begin{aligned}
T_5 = (ax)^m(bx)^{n-2}(cx)^{p-1}\{&(dx)^{s-3}(ex)^t(dc)(db)^2 + (dx)^{s-2}(ex)^{t-1}(ec)(db)^2 \\
&+ (dx)^{s-1}(ex)^{t-2}(ec)(eb)(db) + (dx)^{s-2}(ex)^{t-1}(dc)(db)(eb) \\
&+ (dx)^{s-1}(ex)^{t-2}(dc)(eb)^2 + (dx)^s(ex)^{t-3}(ec)(eb)^2\},
\end{aligned}$$

$$\begin{aligned}
T_6 = (ax)^{m-1}(bx)^{n-1}(cx)^{p-1}\{&(dx)^s(ex)^{t-3}(ea)(eb)(ec) \\
&+ (dx)^{s-1}(ex)^{t-2}(ea)(eb)(dc) + (dx)^{s-1}(ex)^{t-2}(ea)(db)(ec) \\
&+ (dx)^{s-2}(ex)^{t-2}(ea)(db)(dc) + (dx)^{s-2}(ex)^{t-1}(da)(eb)(dc) \\
&+ (dx)^{s-1}(ex)^{t-2}(da)(eb)(ec) + (dx)^{s-2}(ex)^{t-1}(da)(db)(ec) \\
&+ (dx)^{s-3}(ex)^t(da)(db)(dc)\},
\end{aligned}$$

$$\begin{aligned}
T_7 = (ax)^{m-2}(bx)^n(cx)^{p-1}\{&(dx)^{s-3}(ex)^t(da)^2(dc) + (dx)^{s-2}(ex)^{t-1}(da)^2(ec) \\
&+ (dx)^{s-1}(ex)^{t-2}(ec)(da)(ea) + (dx)^{s-2}(ex)^{t-1}(dc)(da)(ea) \\
&+ (dx)^{s-1}(ex)^{t-2}(dc)(ea)^2 + (dx)^s(ex)^{t-3}(ec)(ea)^2\},
\end{aligned}$$

$$T_8 = (ax)^{m-1}(bx)^n(cx)^{p-2}\{(dx)^s(ex)^{t-3}(ea)(ec)^2 + (dx)^{s-1}(ex)^{t-2}(da)(ec)^2 + (dx)^{s-2}(ex)^{t-1}(da)(ec)(dc) + (dx)^{s-1}(ex)^{t-2}(ea)(ec)(dc) + (dx)^{s-2}(ex)^{t-1}(ea)(dc)^2 + (dx)^{s-3}(ex)^t(da)(dc)^2\},$$

$$T_9 = (ax)^m(bx)^{n-1}(cx)^{p-2}\{(dx)^{s-3}(ex)^t(db)(dc)^2 + (dx)^{s-2}(ex)^{t-1}(eb)(dc)^2 + (dx)^{s-1}(ex)^{t-2}(eb)(dc)(ec) + (dx)^{s-2}(ex)^{t-1}(db)(dc)(ec) + (dx)^{s-1}(ex)^{t-2}(db)(ec)^2 + (dx)^s(ex)^{t-3}(eb)(ec)^2\},$$

$$T_{10} = (ax)^m(bx)^n(cx)^{p-3}\{(dx)^s(ex)^{t-3}(ec)^3 + (dx)^{s-1}(ex)^{t-2}(ec)^2(dc) + (dx)^{s-2}(ex)^{t-1}(ec)(dc)^2 + (dx)^{s-3}(ex)^t(dc)^3\}.$$

These are the 56 terms of $\{(ax)^m(bx)^n(cx)^p,\ (dx)^s(ex)^t\}^3$ so arranged that each term is adjacent to those on either side of it, the numerical coefficients being omitted throughout.

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